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Abstract

This paper considers a new class of first order moving average type time series model with index δ (> 0) to describe some hidden features of a time series. It is shown that this class of models provides a valid, simple solution to a new direction of time series modelling. In particular, for suitably chosen parameters (coefficient β and index δ) this type of models could be used to describe data with low or high frequency components. Various new results associated with this class are given in a general form. A simulation study is carried out to justify the theory. We justify the importance of this class of models in practice using a set of real time series data

***Keywords*:** Time series, Misclassification, High frequency, Spectrum, Estimation, Financial data, Moving average, Correlation, Index, Periodogram

1 INTRODUCTION

It is known that there are some problems arise in time series modelling in practice for data with high frequency components, for example, financial data. Although ARMA type models could be used in practice, there is no systematic approach or a suitable class of time series models available in literature to accommodate, analyze and forecast of time series with changing frequency behaviour via a direct method. This paper attempts to introduce a family of first order moving average (MA(1)) type models (Generalized MA(1) or GEMA(1)) to describe some hidden properties of time series data.

We first consider the family of standard $MA(1)$ processes (see, for instance, Box and Jenkins (1976), Priestley (1989), Brockwell and Davis (1991)) generated by

$$X_t = Z_t - \beta Z_{t-1}\,, \tag{1.1}$$

where $|\beta| < 1$ and $\{Z_t\}$ is a sequence of uncorrelated random variables (not necessarily independent) with zero mean and variance σ^2 , known as white noise, $WN(0, \sigma^2)$.

Using the backshift operator, B (i.e. $B^j X_t = X_{t-j},\ j \geq 0$) and the identity operator $I = B^0$, (1.1) can be written as

$$X_t = (I - \beta B) Z_t\,. \tag{1.2}$$

The process in (1.2) have the following properties:

(i) The autocorrelation function (acf), ρ_k satisfies
$\rho_1 = -\beta/(1+\beta^2)$ and $\rho_k = 0$ for $k \geq 2$,

(ii) The partial autocorrelation function (pacf), ϕ_k satisfies,
$\phi_k = -\beta^k(1-\beta^2)/(1-\beta^{2(k+1)})$,

(iii) The spectral density function (sdf), $f_X(\omega)$ is

$$f_X(\omega) = \frac{\sigma^2}{2\pi}(1 - 2\beta\, Cos\, \omega + \beta^2);\ -\pi \leq \omega \leq \pi. \tag{1.3}$$

It is clear from the above that for $0 < \beta < 1$, ρ_1 is negative and the series fluctuate rapidly about its mean. The spectrum is dominated by high frequencies and this behaviour is represent by a large peak of the spectrum near the frequency $\omega = \pi$. In other words the power (of the process) is concentrated at high frequencies (the maximum value of $f_X(\omega)$ occurs at $\omega = \pi$ and this is given by $f_X^s(0) = \frac{\sigma^2}{2\pi}(1+\beta)^2$, where s stands for standard). Also note that $|\phi_k| < \beta$, and the pacf is dominated by a damped exponential.

It is well known that for any time series data set, the density of crossings at a certain level $x_t = x$ may vary. We have noticed that this property (say, the variation in the degree of frequency) is very common in many time series data sets (see the graphs 1 to 4 in the Appendix I). However, these series cannot discriminate from each other using the standard time series techniques. In other words the acf, the pacf, and the spectrum are similar to each other in some cases and one may propose the same model for all of these cases. Although these models provide a valid basic solution, a complete picture of the underlying process can not be achieved by using these standard models. This motivates us to answer the following question: *'How can we indicate the degree of frequency in time series modelling'?* This leads us to introduce a generalized version of (1.2) with an additional parameter (or index) $\delta(> 0)$ (to control the degree of frequency) satisfying

$$X_t = (I - \beta B)^{\delta} Z_t;\ 0 < \beta < 1;\ \delta > 0\,. \tag{1.4}$$

This class of models covers the traditional MA(1) family given in (1.1) when $\delta = 1$. It is interesting to note that the frequency of data can be controlled by this additional parameter δ and hence (1.4) constitutes a wide variety of important processes in practice. There are a large number of real world data sets with varying frequencies (especially in finance there are data with high frequency) and (1.4) can be applied easily using existing techniques with some modifications. The class of models generated by (1.4) is called **'generalized MA(1)'** or **'GEMA(1)'**. Although (1.4) can easily be extended to general $ARMA$ type models, this paper considers the family of $GEMA(1)$ given in (1.4) with $0 < \beta < 1$ and $\delta > 0$, where δ is a real number.

With that view in mind Section 2 reports some properties of the underlying process in (1.4).

2 Properties of $GEMA(1)$ Processes

Let

$$(I - \beta B)^{\delta} = \sum_{j=0}^{\infty} \psi_j \beta^j B^j\,, \tag{2.1}$$

where $0 < \beta < 1,\ \delta \in \mathcal{R}^+;\ B^0 = I,\ \psi_0 = 1$ and

$$\begin{aligned} \psi_j &= (-1)^j \binom{\delta}{j} \\ &= \frac{(-\delta)(-\delta+1)\cdots(-\delta+j-1)}{j!};\ j \geq 1 \end{aligned}$$

If δ is a positive integer, then $\psi_j = 0$ for $j \geq \delta + 1$. For any non-integral $\delta > 0$, it is known that,

$$\psi_j = \frac{\Gamma(j-\delta)}{\Gamma(j+1)\Gamma(-\delta)}, \tag{2.2}$$

where $\Gamma(\cdot)$ is the gamma function given by

$$\Gamma(x) = \begin{cases} \int_0^\infty t^{x-1} e^{-t} dt\ ; & x > 0 \\ \infty\ ; & x = 0 \\ x^{-1}\Gamma(x+1)\ ; & x < 0\,. \end{cases}$$

It is easy to see that the series $\sum_{j=0}^\infty \psi_j \beta^j$ converges for all δ since $|\beta| < 1$, and in particular, the process X_t in (1.4) is equivalent to a valid $MA(\infty)$ process of the form $X_t = \sum_{j=0}^\infty \psi_j \beta^j Z_{t-j}$ with

$$\sum |\psi_j \beta^j|^2 < \infty. \tag{2.3}$$

Now we state and prove the following theorem for a stationary solution of (1.4).

Theorem 2.1: Let $\{Z_t\} \sim WN(0, \sigma^2)$. Then for all $\delta > 0$ and $|\beta| < 1$, the infinite series

$$X_t = \sum_{j=0}^\infty \psi_j \beta^j Z_{t-j}, \tag{2.4}$$

converges absolutely with probability one, where Ψ_j is given in (2.2).

Proof: Using the facts $E\left(\sum_{j=0}^\infty |\psi_j \beta^j Z_{t-j}|\right)^2 = \sum_{j=0}^\infty |\psi_j \beta^j|^2\ E\{|Z_{t-j}|^2\}$.

and $\sum_{j=0}^\infty |\psi_j \beta^j|^2 < \infty$, the result follows.
Thus (2.4) gives a stationary solution for the GEMA(1) model in (1.4).

Note: For $\beta = 1$, (2.4) converges for all δ in $0 < \delta < 1/2$.

It is clear from (1.4) that $\{X_t\}$ is equivalent to a valid AR(∞) process of the form

$$\sum_{j=0}^{\infty} \pi_j \, \beta^j \, X_{t-j} = Z_t, \tag{2.5}$$

$$\begin{aligned} \text{where } \pi_j &= (-1)^j \binom{-\delta}{j} \\ &= \frac{\Gamma(j+\delta)}{\Gamma(j+1)\,\Gamma(\delta)}; \;\; j \geq 0. \end{aligned}$$

Let $\gamma_k = Cov(X_t, \, X_{t-k}) = E(X_t \, X_{t-k})$ be the autocovariance function (at lag k) of $\{X_t\}$ satisfying the conditions of theorem 2.1. It is clear from (2.5) that $\{\gamma_k\}$ satisfy a Yule-Walker type recursion

$$\sum_{j=0}^{\infty} \pi_j \, \beta^j \, \gamma_{k-j} = 0 \, ; \;\; k > 0 \tag{2.6}$$

and the corresponding autocorrelation function (acf), ρ_k, at lag k is given by

$$\sum_{j=0}^{\infty} \pi_j \, \beta^j \rho_{k-j} = 0 \, ; \;\; k > 0 \, , \tag{2.7}$$

The spectrum (sdf) of $\{X_t\}$ in (1.4) is

$$\begin{aligned} f_X(\omega) &= |1 - \beta \, e^{-i\omega}|^{2\delta} \, \frac{\sigma^2}{2\pi}; \;\; -\pi \leq \omega \leq \pi \\ &= (1 - 2\beta \, Cos \, \omega + \beta^2)^{\delta} \, \frac{\sigma^2}{2\pi} \, . \end{aligned} \tag{2.8}$$

Note: In a neighbourhood of $\omega = \pi$,

$$f_X^g \sim \frac{\sigma^2}{2\pi}(1+\beta)^{2\delta},$$

where g stands for generalized. For $\delta > 1$ and $0 < \beta < 1$, it is clear (from (1.3) and (2.8)) that $f_X^g > f_X^s$. Thus a set of high frequency data satisfying an $MA(1)$ can be replaced by an $GEMA(1)$ model with a suitably chosen index δ.

In Section 3 we obtain the exact form of γ_k (or ρ_k) using the sdf in (2.8). That is,

$$\begin{aligned} \gamma_k &= \int_{-\pi}^{\pi} e^{ik\omega} \, f_X(\omega) d\omega \\ &= \frac{\sigma^2}{\pi} \int_0^{\pi} Cos(k\omega)(1 - 2\beta \, Cos \, \omega + \beta^2)^{\delta} \, d\omega. \end{aligned} \tag{2.9}$$

We first evaluate the integral in (2.9) for $k = 0$ and then for other values of $k \geq 1$.

3 Main Results

Theorem 3.1:

$$\gamma_0 = Var(X_t) = \sigma^2 F(-\delta,\, -\delta;\; 1;\; \beta^2) \tag{3.1}$$

where $F(\theta_1, \theta_2;\; \theta_3;\; \theta)$ is the hypergeometric function given by

$$F(\theta_1, \theta_2;\; \theta_3;\; \theta) = \sum_{j=0}^{\infty} \frac{\Gamma(\theta_1 + j)\, \Gamma(\theta_2 + j)\, \Gamma(\theta_3)\, \theta^j}{\Gamma(\theta_1)\, \Gamma(\theta_2) \Gamma(\theta_3 + j)\, \Gamma(j+1)}\,. \tag{3.2}$$

Note that the right hand side of (3.2) terminates if θ_1 or θ_2 is equal to a negative integer (see also Gradsteyn and Ryzhik (GR)(1965), p.1039).

Proof: From GR, p.384 (3.665:2),

$$\int_0^{\pi} \frac{d\omega}{(1 - 2\beta\, Cos\omega + \beta^2)^{\delta'}} = B(\frac{1}{2}, \frac{1}{2}) F(\delta', \delta';\; 1;\; \beta^2), \tag{3.3}$$

where $B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$ is the Beta function.
Since $B(\frac{1}{2}, \frac{1}{2}) = \pi$, the result follows since $\delta' = -\delta$.
Using (3.2), it is easy to see that for $\delta' = -1$,

$$F(-1, -1;\, 1;\, \beta^2) = (1 + \beta^2).$$

Hence (3.1) confirms the corresponding well known result for the variance of an MA(1) (standard) process satisfying (1.2) with $\delta = 1$. That is,

$$Var(X_t) = \sigma^2 (1 + \beta^2),\;\; |\beta| < 1.$$

As it is not easy to evaluate the integral in (2.10) for $k \neq 0$, we find an expression for γ_k via ,

$$\begin{aligned} \gamma_k &= E(X_t\, X_{t-k}) \\ &= \sigma^2 \sum_{j=0}^{\infty} \psi_j\, \psi_{j+k}\, \beta^{k+2j}\,. \end{aligned}$$

An explicit form of γ_k is given in Theorem 3.2 .

Theorem 3.2:

$$\gamma_k = \frac{\sigma^2\, \beta^k \Gamma(k-\delta)\, F(-\delta,\, k-\delta;\, k+1;\, \beta^2)}{\Gamma(-\delta)\,\Gamma(k+1)};\;\; k \geq 0\,. \tag{3.4}$$

Proof: Since $X_t = \sum_{j=0}^{\infty} \psi_j\, \beta^j\, Z_{t-j}$,

$$\begin{aligned}\gamma_k &= \sigma^2\, \beta^k \sum_{j=0}^{\infty} \psi_j\, \psi_{j+k}\, \beta^{2j}, \\ &= \sigma^2 \beta^k \sum_{j=0}^{\infty} \frac{\Gamma(j-\delta)\,\Gamma(j+k-\delta)(\beta^2)^j}{\Gamma^2(-\delta)\Gamma(j+1)\Gamma(j+k+1)}\,.\end{aligned}$$

From p.556 of Abramovitz & Stegun (1965), we have

$$\sum_{j=0}^{\infty} \frac{\Gamma(-\delta+j)\,\Gamma(k-\delta+j)(\beta^2)^j}{\Gamma(k+1+j)\,\Gamma(j+1)} = \frac{\Gamma(-\delta)\,\Gamma(k-\delta)\, F(-\delta,\, k-\delta;\, k+1;\, \beta^2)}{\Gamma(k+1)}$$

and hence (3.4) follows.

Note: When $k = 0$, Theorem 3.2 reduces to Theorem 3.1 .

The autocorrelation function plays an important role in the analysis of the underlying process. The Corollary 3.1 below gives ρ_k for any $k \geq 0$.

Corollary 3.1 : The autocorrelation function of the GEMA(1) process in (1.4) is

$$\rho_k = \beta^k \frac{\Gamma(k-\delta)\, F(-\delta,\, k-\delta;\, k+1;\, \beta^2)}{\Gamma(-\delta)\,\Gamma(k+1)\, F(-\delta, -\delta;\, 1;\, \beta^2)} \tag{3.5}$$

Note: It is interesting to note that (3.5) reduces to the acf of a standard $AR(1)$ process satisfying $X_t = \alpha X_{t-1} + Z_t$ when $\delta = -1$ (see Peiris (2002)), since

$$F(1, k+1;\, k+1;\, \alpha^2) = F(1, 1;\, 1;\, \alpha^2) = (1-\alpha^2)^{-1}$$

(see also GR p.1040).

Thus the results of these two theorems provide a new set of formulae in general form. Obviously, our new result for γ_k in (3.4) supersede all existing acf for standard $AR(1)$ $MA(1)$ and also for fractionally differenced white noise processes.

Remark : An important consequence of our new result of Theorems 3.2 yield

$$\int_0^{\pi} \frac{Cos\,k\,\omega\;d\,\omega}{(1-2\beta\,Cos\,\omega+\beta^2)^{\delta'}} = \frac{\pi\beta^k\,\Gamma(k+\delta')\,F(\delta',\,k+\delta';\,k+1;\,\beta^2)}{\Gamma(\delta')\,\Gamma(k+1)} \tag{3.6}$$

When $k = 0$ the equation (3.6) reduces to (3.3) in Theorem 3.1 . (also see GR p 384).

Note : The new result in equation (3.6) is, particularly, useful in many theoretical developments of generalized ARMA (GARMA) processes with indices and this will be discussed inna future paper.

Now in Section 4, we discuss a method of estimating parameters β and δ appeared in (1.4).

4 Estimation of Parameters

Consider the ratio of ρ_{k+1}/ρ_k.

From Corollary 3.1 (equation 3.7), we have

$$\frac{\rho_{k+1}}{\rho_k} = \frac{\alpha(k-\delta)F(-\delta,k+1-\delta;k+2;\beta^2)}{(k+1)F(-\delta,k-\delta;k+1;\beta^2)}. \tag{4.1}$$

From the properties of $F(\theta_1,\theta_2;\theta_3;\theta)$ given in equation (3.2), it is not difficult to show that the right hand side (rhs) of (4.1) is approximately equal to β for large k.

Thus an estimate of β is obtained by

$$\hat{\beta} = \frac{r_{k+1}}{r_k}, \tag{4.2}$$

where r_k is the lag k sample acf.

However, the following approximations for ρ_1 and ρ_2 will produce the method of moment (MOM) estimates for both β and δ. that is,

$$\rho_1 = -\beta\delta, \tag{4.3}$$

and

$$\rho_2 = \frac{-\delta(1-\delta)\beta^2}{2}. \tag{4.4}$$

The corresponding MOM are given by

$$\hat{\beta} = 2r_2/r_1 - r_1, \hat{\delta} = -r_1/\hat{\beta}.$$

A better approximation for ρ_1 is given by $\rho_1 = -\beta\delta(1-\beta^2\delta^2)$. The maximum likelihood estimation (MLE) is also possible and will be discussed in a future paper.

Now one can modify the usual regression approach (via periodogram analysis) as in the long memory case (see Geweke and Porter-Hudak (1983), Brockwell and Davis (1991), Peiris and Court (1993), Chen et. al. (1994), Hunt et. al. (2001)) to estimate δ.

From (2.8) we have

$$ln f_X(\omega) = C - \delta\, ln(|1-\beta e^{-i\omega}|^2), \tag{4.5}$$

where $C = ln(\sigma^2/2\pi)$. Suppose we have T observations $X_1, X_2, \cdots, X_T$. The corresponding sample periodogram is

$$I_{T,X}(\omega_j) = \frac{1}{2\pi}\sum_{h=-(T-1)}^{T-1} \hat{\gamma}_h e^{-ih\omega_j},$$

where $\hat{\gamma}_h = \frac{1}{T}\sum_{j=1}^{T-|h|}(X_{j+|h|}-\bar{X})(X_j-\bar{X})$ and $\omega_j = 2\pi j/T$.

Now the equation in (4.6) reduces to the linear regression equation

$$y_j = a - \delta x_j + \varepsilon_j, \quad j = 1, \cdots, K_T,$$

where $y_j = ln\{I_{T,X}(\omega_j)\}$, $x_j = ln(|1-\hat{\beta}e^{-i\omega_j}|^2)$, $a = C$, $\varepsilon_j = ln(\frac{I_{T,X}(\omega_j)}{f_X(\omega_j)})$ and K_T is a constant chosen so that $2\pi K_T/T$ is small. In practice we select $K_T = T^\eta, 0 < \eta < 1$.

Thus an estimator for δ is

$$\hat{\delta}_p = -\left[\sum_{j=1}^{K_T}(x_j-\bar{x})y_j\right] \Bigg/ \left[\sum_{j=1}^{K_T}(x_j-\bar{x})^2\right]. \tag{4.6}$$

Since $var(\varepsilon_j) = \frac{\pi^2}{6}$, the asymptotic distribution of $\hat{\delta}$ is

$$\hat{\delta} \sim N(\delta, \frac{\pi^2}{6\sum(x_j-\bar{x})^2}).$$

Peiris and Court (1993), Chen et al (1994), and Hunt et al (2001) suggested replacing the periodogram in the above expression by a general scaled lag window estimator for $f_X(\omega)$. Suppose that $\kappa(x)$ is a real valued, bounded symmetric function defined and continuous on $x \in [-1,1]$, and 0 elsewhere. Let $\{R_T\}$ be a sequence of integers such that $R_T \to \infty$ and $R_T/T \to 0$ as $T \to \infty$. The general form of a scaled lag window estimator for $f_X(\omega)$ is

$$\hat{f}_X(\omega) = \frac{1}{2\pi} \sum_{|r|<T} \kappa\left(\frac{r}{R_T}\right) C_r \cos(\omega r), \tag{4.7}$$

where

$$C_r = \frac{1}{T-|r|} \sum_{t=1}^{T-|r|} (X_t - \mu)(X_{t+|r|} - \mu), \quad |r| = 0, 1, \cdots, T-1$$

is an unbiased estimator of the autocovariance function of $\{X_t\}$, γ_r , calculated knowing the true mean μ . The lag window estimator for d is obtained by substituting $\hat{f}_X$ for $I_{T,X}$ in (4.4). Denote the resulting estimator by $\hat{\delta}_L$. That is,

$$\hat{\delta}_L = -\left[\sum_{j=1}^{K_T}(x_j - \bar{x})y_j\right] \Big/ \left[\sum_{j=1}^{K_T}(x_j - \bar{x})^2\right], \tag{4.8}$$

where $y_j = ln\hat{f}_X(\omega_j)$.

Further, let

$$C_r^* = \frac{1}{T} \sum_{t=1}^{T-|r|} (X_t - \bar{X})(X_{t+|r|} - \bar{X}), \quad |r| = 0, \cdots, T-1$$

be an estimator of γ_r calculated using the sample mean $\bar{X}$. The corresponding spectral density estimate using C_r^* is denoted by

$$\hat{f}_X^*(\omega) = \frac{1}{2\pi} \sum_{|r|<R_T} \kappa\left(\frac{r}{R_T}\right) C_r^* \cos(\omega r). \tag{4.9}$$

Let $\hat{\delta}_L^*$ denote the estimator obtained by substituting $\hat{f}_X^*$ for $I_{T,X}$ in (4.4).

Chen, Abraham, and Peiris (1994) investigated the bias and mean squared error of $\hat{\delta}_L$ via some simulation studies using various lag window spectral density estimators. They noted that while $\hat{\delta}_L$ had smaller mean squared error compared with $\hat{\delta}_p$ the estimator did typically have larger bias. In Section 5 we apply this theory to a real data set given in Abraham and Ledolter (1983).

5 An Application

Consider the time series of 197 readings from a chemical process concentration (in every two hours) given in series A of Box and Jenkins (1976). The original series is nonstationary. The time series plot, the acf, the pacf and the spectrum indicate that the differenced data has a high frequency component. Since the original series is nonstationary, Box and Jenkins (1976) fitted the following MA(1) model for the differenced data:

$$x_t = z_t - 0.7z_{t-1}, \tag{5.1}$$

where x_t represents the differenced data.

However, we fit a generalized MA(1) model for the data in order to explain the high frequency behaviour using the methods described in Section 4. The results are:

$\hat{\beta} = 0.414$, $\hat{\delta} = 0.620$, and $\sigma^2 = 0.138$. The fitted model is

$$x_t = (I - 0.414)^{0.620} z_t, \tag{5.2}$$

where $var(z_t) = 0.138$.

The first three forecasts from the time origin at $t = 193$ and the corresponding 95% confidence intervals for x_t are:

0.02 and 0.02 ± 0.52

-0.05 and -0.05 ± 0.30

0.005 and 0.005 ± 0.38

The true (observed) values for the last three readings on x_t are -0.10, -0.50, and 0.20 respectively.

The corresponding results due to standard MA modelling are:

-0.10 and -0.10 ± 0.79

0.00 and 0.00 ± 0.87

0.00 and 0.00 ± 0.87

Our results are closer to the true values than in the traditional MA (or ARMA) modelling. Also note that our new results provide shorter confidence intervals in all three cases above (see Appendix II for comparison).

Note: All calculations and simulations reported here are carried out using Splus.

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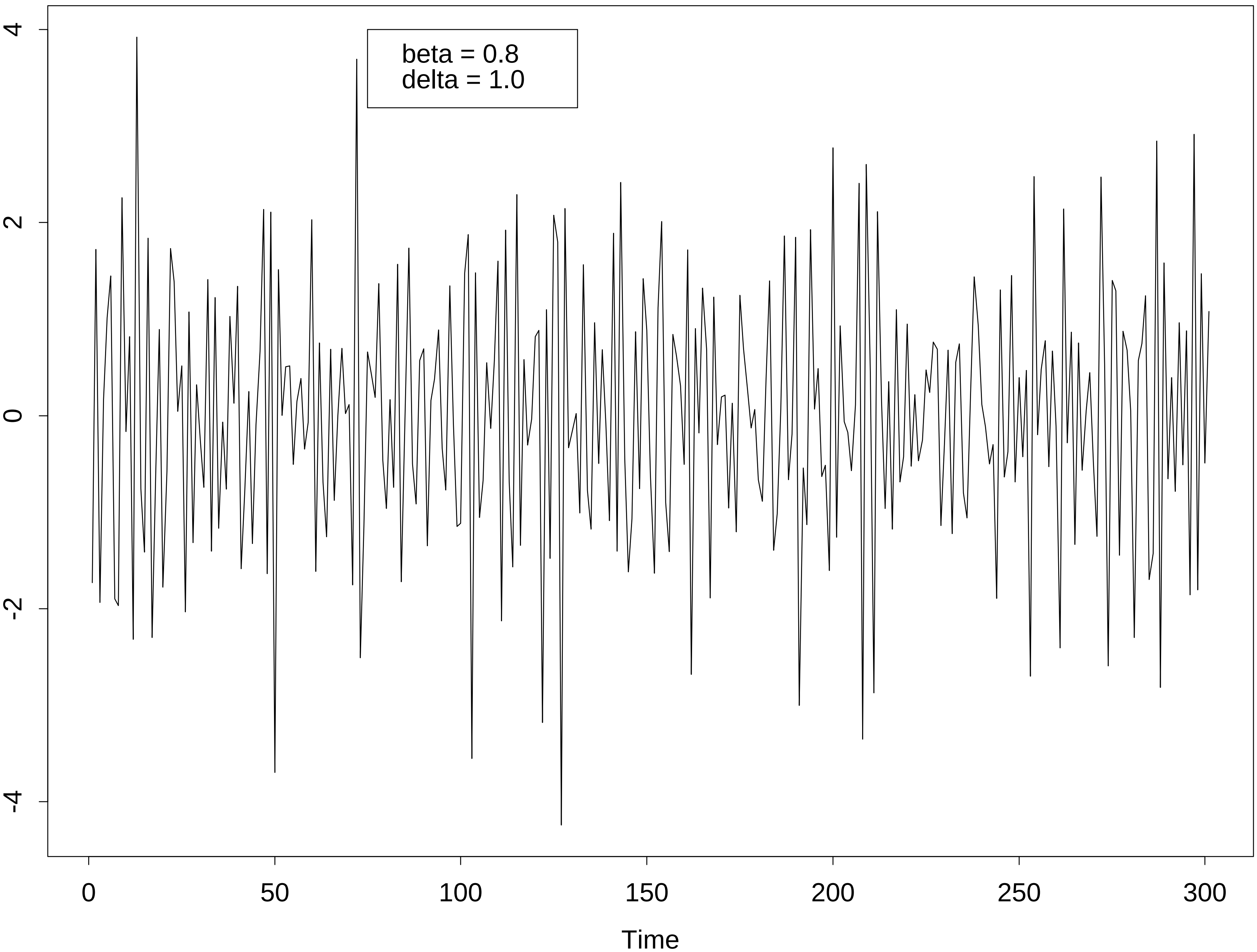
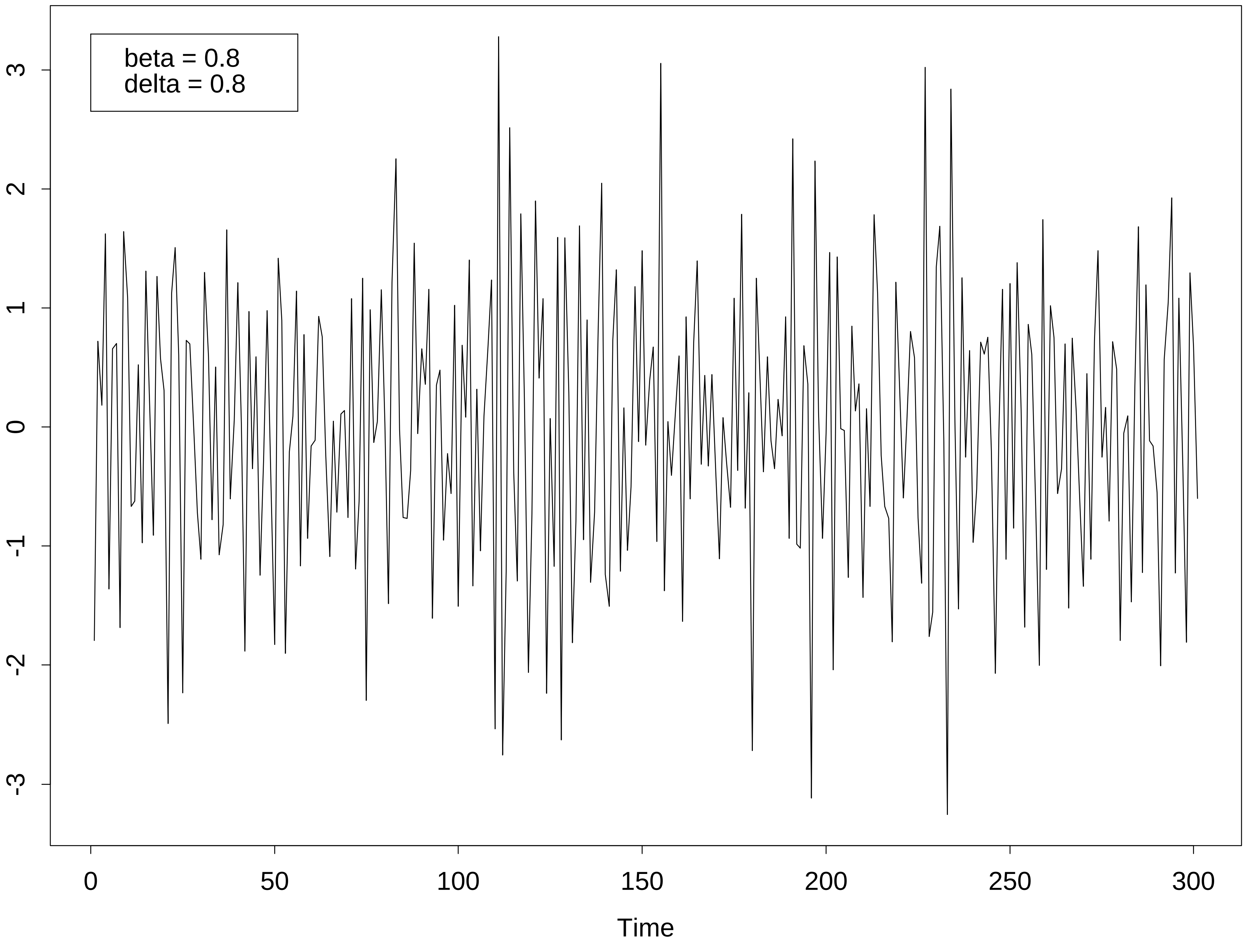
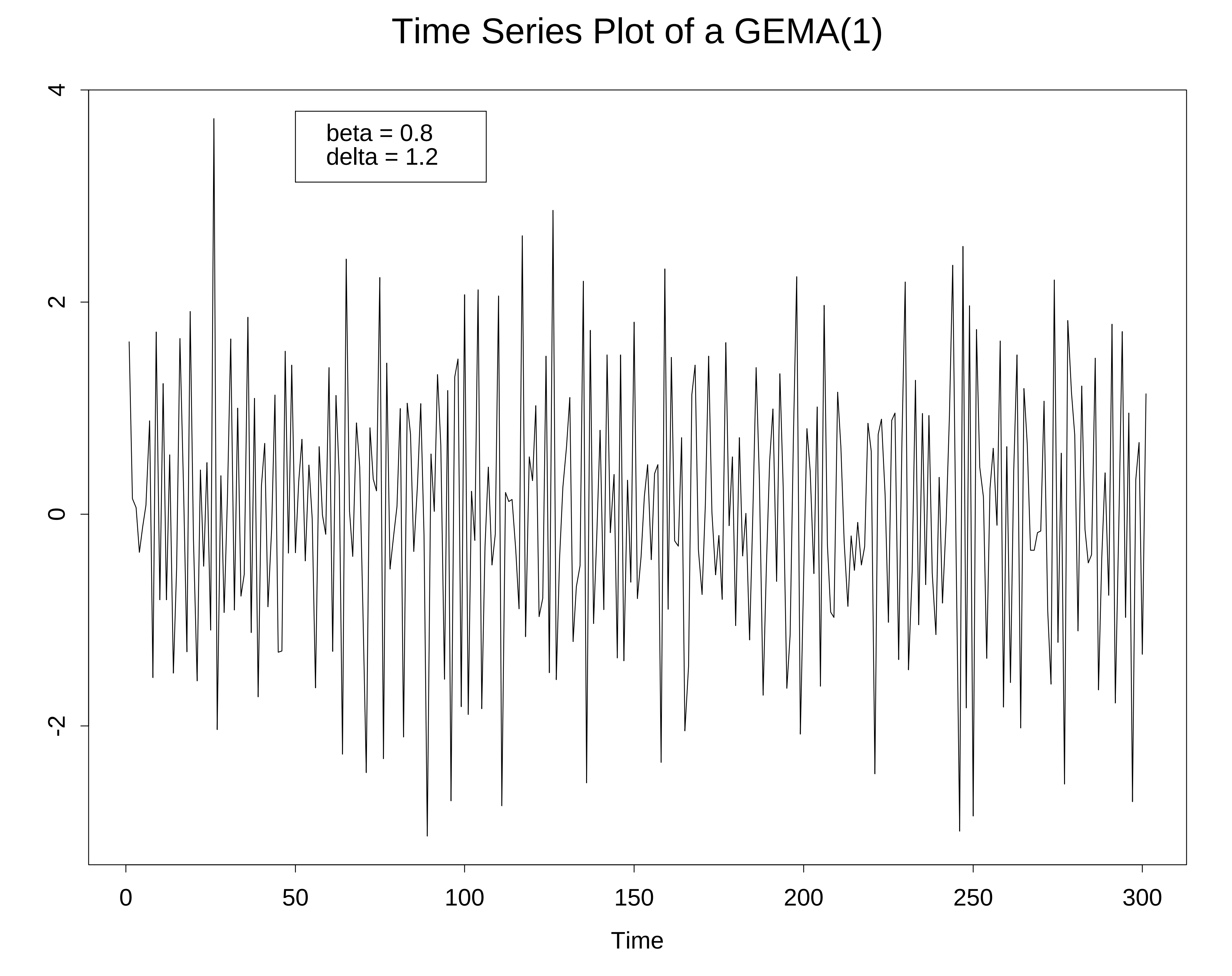
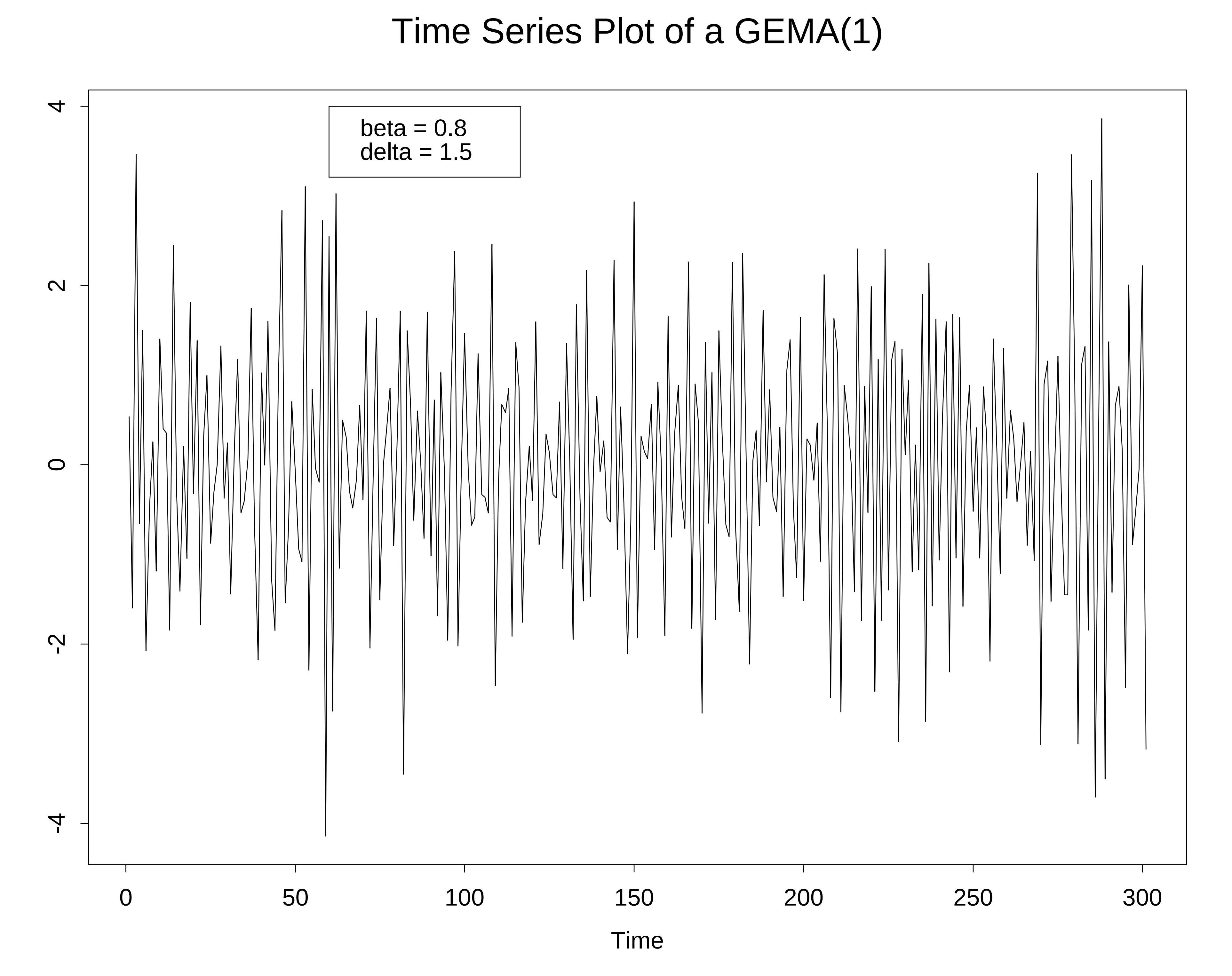
Time Series Plot of an MA(1)

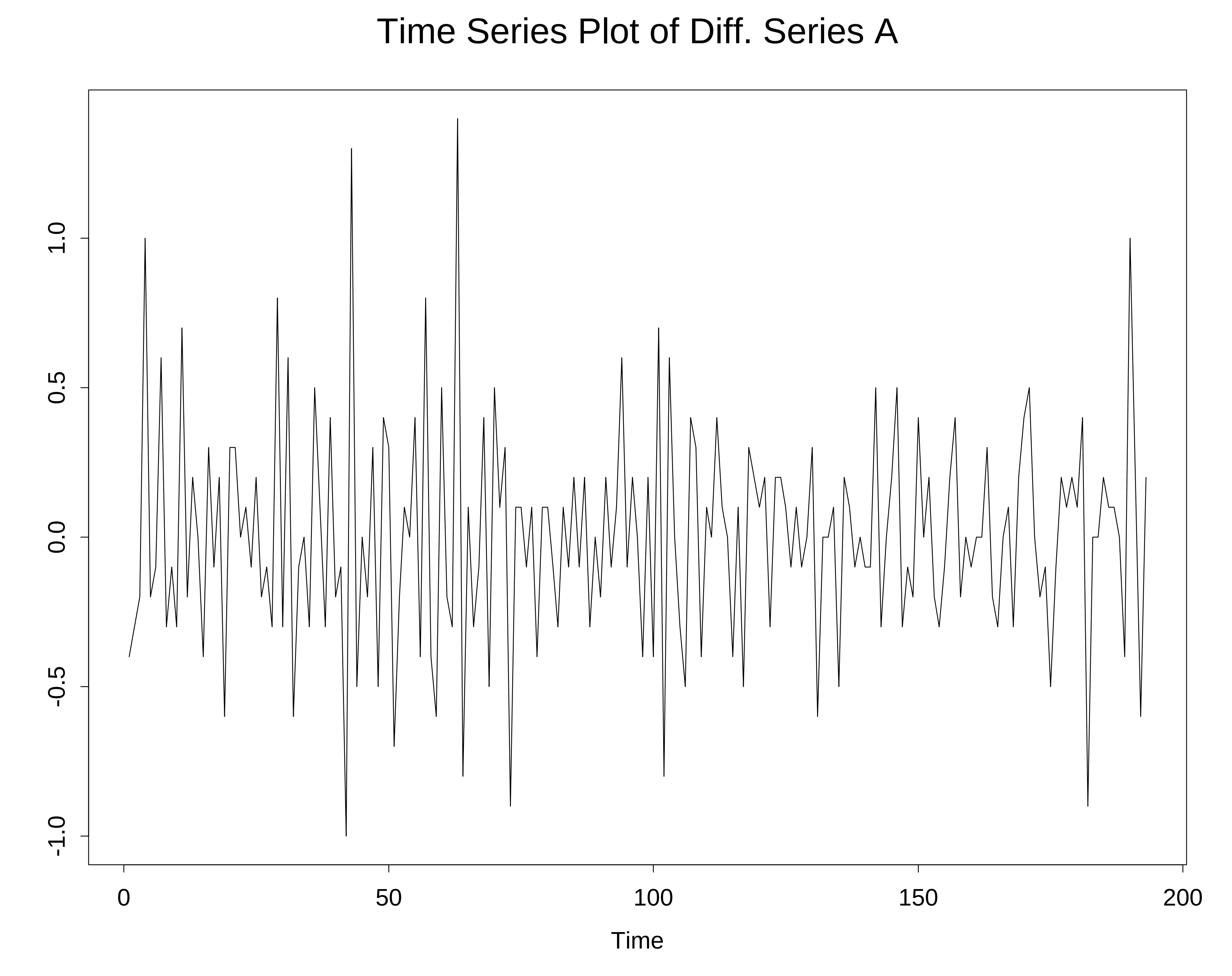


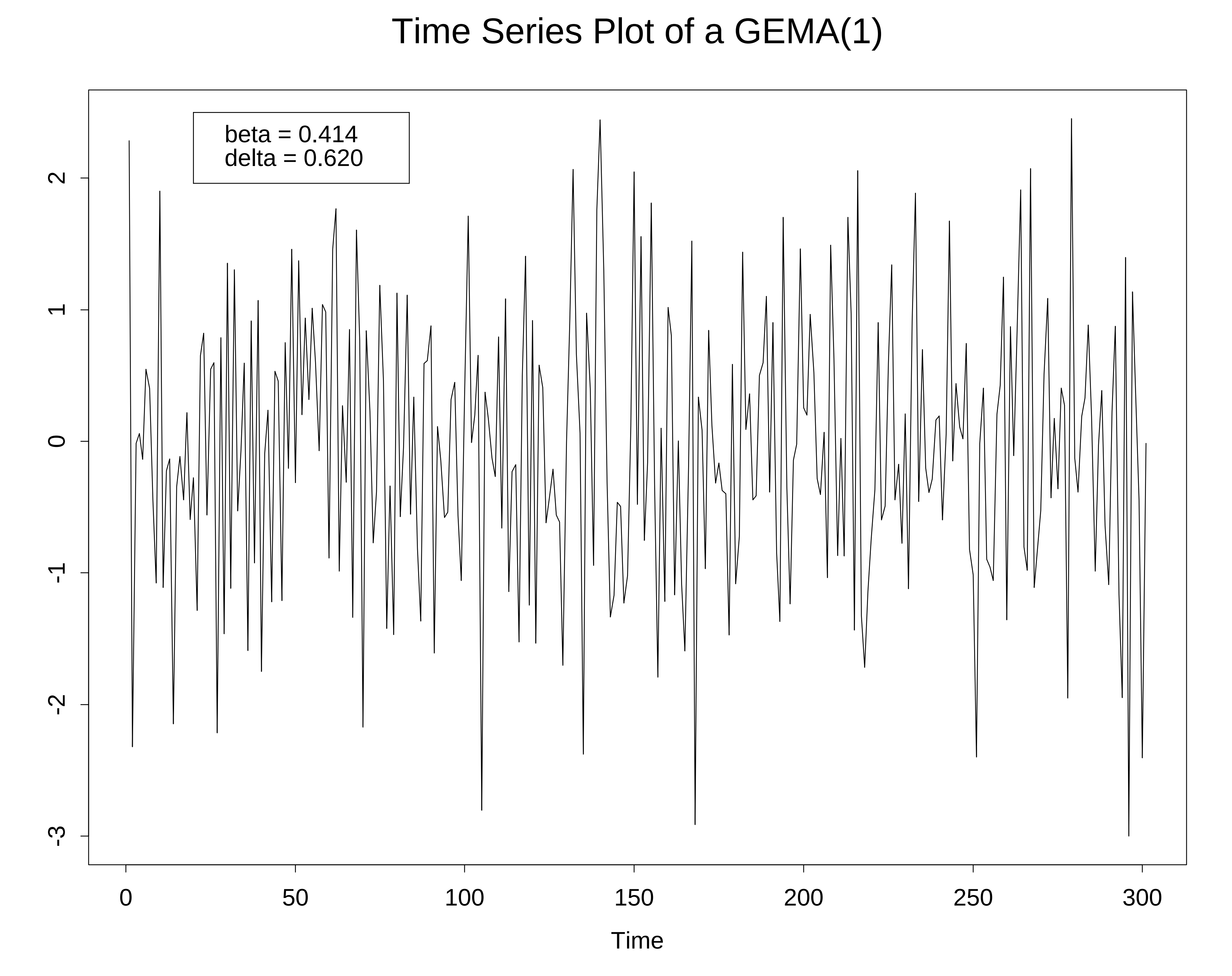
Time Series Plot of a GEMA(1)




Time Series Plot of a GEMA(1)
beta = 0.8
delta = 1.2
4
2
0
-2
0
50
100
150
200
250
300
Time


Time Series Plot of a GEMA(1)
beta = 0.8
delta = 1.5
4
2
0
-2
-4
0
50
100
150
200
250
300
Time


Time Series Plot of Diff. Series A
1.0
0.5
0.0
-0.5
-1.0
0
50
100
150
200
Time


Time Series Plot of a GEMA(1)
beta = 0.414
delta = 0.620
2
1
0
-1
-2
-3
0
50
100
150
200
250
300
Time