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Mathematical connections established in the teaching of functions

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This study explores the types of mathematical connections established in the classroom in the teaching of functions. An extended model for mathematical connections (different representations (DR), procedural (PC), if-then (I-T), part-whole connections (PWC), feature/property (F/P), analogies, and instruction-oriented connections (IOC)) is used as the analytical framework. The context for the study is classroom observations of two secondary mathematics teachers teaching functions to Grade 9 students. The strength of teachers' mathematical knowledge for teaching (MKT) the concept of function is different: one has stronger MKT, while the other has weaker MKT. A total of 485 connections are identified in a sample of 24 ninth-grade lessons observed (12 lessons per teacher). The teacher with stronger MKT produces far more connections ($f = 317$) than the teacher with weaker MKT ($f = 168$), and she mostly establishes I-T, DR, PWC and F/P type of connections. The teacher with weaker MKT frequently makes procedural types of connections. This 'connections gap' may reflect differences in the teachers' MKT and in their beliefs about the teaching and learning of mathematics. The study also documents some of the important internal connections within functions based on the observed lessons, and an additional IOC has emerged from the data.

1. Introduction

A mathematical connection is an exact relationship between two or more mathematical ideas (Businskas, 2008). Two broad typologies are suggested: intra-mathematical connections (the focus of this study) and extra-mathematical connections. *Intra-mathematical connections* are formed between ideas, concepts, theorems, procedures or representations in mathematics, while *extra-mathematical connections* are established between mathematical concepts or models and problems in non-mathematical contexts, or vice versa (Gamboa *et al.*, 2021). There is a consensus among the mathematics education community on the importance of making connections in the teaching and learning of mathematics (e.g., Coxford, 1995; Toh & Choy, 2021). Also, there has been an increased emphasis for mathematical connections in national curricula in many countries such as the USA (National Council of Teachers of Mathematics [NCTM], 2010, 2014), Singapore (Ministry of Education [MOE], 2012, 2018), Australia

(Australian Curriculum, Assessment and Reporting Authority [ACARA], 2018) and Turkey (Turkish Board of Education, 2011, 2018). Students develop key competencies including linking conceptual and procedural knowledge, recognising equivalent representations of the same concept, using the connections across mathematical topics and seeing mathematics as an integrated whole, and applying mathematical modelling to solve problems that arise in other disciplines as a result of experiencing connections in mathematics (Coxford, 1995; Low & Wong, 2021). However, little is known about what connections are made in everyday teaching contexts in mathematics classrooms. Most of the investigations in this field focus on understanding the concept of mathematical connections (e.g., Businskas, 2008) but not the connections made during teacher–students interactions.

While there is no universal agreement about what constitutes making connections in mathematics teaching, everyone would agree with the idea that a teacher's ability to recognise and produce connections is linked to their knowledge of mathematics (e.g., Gamboa *et al.*, 2020; Toh & Choy, 2021). As part of a larger research exploring the relationships between secondary mathematics teachers' *mathematical knowledge for teaching (MKT)* (Ball *et al.*, 2008) about the concept of function and their students' learning of this concept (Hatisaru, 2014), this study aims to understand the intra-mathematical connections (hereafter used as 'mathematical connections') in the instruction of two secondary mathematics teachers. An extended model for mathematical connections based on the most relevant mathematical connections typologies in this field (Rodríguez-Nieto *et al.*, 2020) is used for the investigation. The study is significant for several reasons. It first extends the literature by identifying the mathematical connections made in the classroom and contributes to the validity of the current extended model for mathematical connections. The study also documents some of the important internal connections within functions found in the observed lessons (see Sections 3.2 and 4.2; Appendix A) and an additional instruction-oriented connection (IOC) that emerged from the data: *revisiting taught/learnt knowledge*.

2. Conceptual basis

2.1. Mathematical connections

A mathematical connection refers to the relationship between two or more mathematical concepts where one is related to the other. It is 'a cognitive process through which a person relates two or more ideas, concepts, definitions, theorems, procedures, representations and meanings among them, with other disciplines or with real life' (García-García & Dolores-Flores, 2018, p. 229). Several researchers have sought to understand the concept of connections and have identified the types. One of the initial conceptual models for investigations in this field was proposed by Evitts (2004). In his study on the types of mathematical connections preservice mathematics teachers made and used while solving problems, Evitts (2004) identified five broad categories: *modelling connections* (the interaction of real-world information with an appropriate mathematical representation), *structural connections* (the sameness of two mathematical ideas or constructs), *representational connections* (mathematical connections among representations such as graphical, numerical, symbolic, pictorial or verbal forms), *procedure-concept connections* (conceptual and procedural knowledge) and *connections between strands of mathematics* (connections among various domains of mathematics that contribute to the conception of mathematics as an integrated whole). In their study on the types of mathematical connections prospective middle school teachers used while completing tasks on making connections, Eli *et al.* (2011) suggested the following types of connections: *categorical* (using surface features when defining a group/category), *procedural* (relating ideas based on a mathematical procedure through construction of an example), *characteristics or property* (defining the concepts or describing properties of them), *derivation* (building a new

Table 1. *Examples of mathematical connections representing each type*

Connection type	Example
PWC	
Inclusion	Functions are specific types of relations, and all functions are relations.
Generalisation	$y = ax^2 - c$ is a generalisation of $y = x^2–4$, or $y = x^2–4$ is a specific case of $y = ax^2—c$.
F/P	A rectangle has two sets of parallel sides and four 90-degree angles.
AC	A function is like a machine with inputs and outputs. An equation is like a balance scale.
DR	
Equivalent representations	$y = (x + 2) \times (x—2)$ is an equivalent representation of $y = x^2–4$.
Alternate representations	The graph of $y = x^2–4$, for instance, is an alternate representation of this function.
I-T	In equation $x^2–4 = 0$, second degree root implies there are two solutions, namely $x = -2$ and $x = +2$.
IOC	
Prerequisite	Factors and multiples are concepts that must be known to understand working with fractions.
Links with prior knowledge	Linking the concept of function with the concept of relations.
Revisiting taught knowledge	Reminding ninth-grade students that 2^3 is not equal to 3^2 as they are taught in earlier grades.
PC	The backtracking method is performed when solving equations. The discriminant of the equation formula is used in finding the roots of the quadratic equation.

concept upon another concept) and *curricular* (linking concepts in terms of impact to the curriculum). Based on task-based interviews with pre-university students investigating the mathematical connections that they made when plotting the graph of derivative and antiderivative functions, García-García and Dolores-Flores (2019) proposed five types of connections in calculus: *procedural*, *reversibility*, *different representations*, *part-whole* and *feature*.

Businskas (2008) explored how secondary mathematics teachers conceptualise mathematical connections as the interface between their content knowledge and pedagogical content knowledge. Based on structured interviews with nine teachers focusing on quadratic functions and equations, she suggested a model of mathematical connections. In this model, seven types of connections are suggested: *alternate representation*, *equivalent representation*, *common features*, *inclusion*, *generalisation*, *implication* and *procedure*. Businskas' model has been one of the most relevant mathematical connections typologies in this field. Recently, Rodríguez-Nieto *et al.* (2020) refined Businskas' model and the existing models mentioned above. Based on observations of lessons on derivatives taught by an experienced mathematics teacher, they suggested an extension of the current types of mathematical connections including analogies. The present study considered the mathematical connections model developed by Businskas (2008) and extended by Rodríguez-Nieto *et al.* (2020). These conceptual references are briefly presented as employed in this study (Table 1). Although they are represented separately, quite often mathematical connections occur simultaneously (Rodríguez-Nieto *et al.*, 2020).

As elaborated by Rodríguez-Nieto *et al.* (2020), *part-whole connections* (PWC) are manifested in two ways: *inclusion* and *generalization. Inclusion* is present when a mathematical concept is contained within another, and *generalisation* appears when a mathematical concept or idea is the generalisation of a particular case. *Feature* or *property (F/P)* connections occur when characteristics of a mathematical concept are depicted, or properties of the concept are described in terms of what makes it similar or different to others. *Analogical connections (AC)* occur when a conceptual relationship between a familiar source domain and an abstract target domain is established. *Different representations (DR)* are identified when the person represents a mathematical concept by using its alternate and/or equivalent representations. An *equivalent representation* is the transformation of representation made in the same representation system (e.g., symbolic-symbolic). An *alternate representation* is the translation of representation made across representation systems (e.g., symbolic-graphical).

Implication or *if-then connections (I-T)* appear when a mathematical concept leads to another mathematical concept using a logical relationship. *Instruction-oriented connections (IOC)* refer to incorporating the relations among mathematical ideas into teaching and learning of mathematics. They can be manifested both when a new concept is *linked to prior knowledge*, and when an understanding of a new concept is dependent on the understanding of other concepts or *prerequisites*. In this research, however, a new category of IOC that is not identified yet emerged from the data: *revisiting taught knowledge*, and these connections occur when students are reminded about what they have been taught or learnt previously. Finally, *procedural connections (PC)* are identified when the person uses rules, formulae or algorithms to complete a mathematical task. Table 1 presents some examples on each of these connection types.

2.2. *Mathematical connections and teacher knowledge and beliefs*

Making connections among mathematical ideas is important for students to develop a better understanding of mathematics (Bossé, 2003; Cai *et al.*, 2014), and teachers play a significant role in helping students make mathematical connections (Low & Wong, 2021). Although they are not necessarily mutually exclusive, mathematical connections are conceptualised from three broad perspectives: mathematical connections as part of a connected discipline, mathematical connections as part of the process of doing mathematics and mathematical connections as products of understanding (see Singletary, 2012). This study focuses on the latter perspective and considers that a teacher's ability to recognise and make connections is linked to their knowledge of mathematics (e.g., Hughes, 2016; Gamboa *et al.*, 2020). For example, Ball *et al.* (2008) consider connections as a part of teachers' awareness of how mathematical ideas and/or concepts are associated throughout school years. Rowland (2013) suggests *connections* (along with *transformation* and *contingency*) as one of the dimensions of their Knowledge Quartet model that refers to ways in which knowledge is brought to the conduct of teaching. This dimension of teacher knowledge includes a teacher's capability to link the concepts and procedures with each other, anticipate hierarchy or complexity and make decisions about sequencing.

The notion of teacher beliefs is a relevant construct to understand what teachers know and how they teach (e.g., Rowland, 2013). Askew *et al.* (1997) explored the knowledge, beliefs and practices of a sample of effective teachers of numeracy (based on their students' learning gains) and found that effective teachers have a particular set of coherent beliefs and understanding, and these underpin their teaching of the content. Effective teachers had knowledge and understanding of links between the areas that they teach. They believed that 'being numerate requires having a rich network of connections between different mathematical ideas', and accordingly, they used instructional approaches that 'connected different areas of mathematics and different ideas in the same area of mathematics using a variety of words, symbols and diagrams' (p. 4). Singletary (2012) examined the role of beliefs about mathematics in the mathematical connections that three mathematics teachers established in their instructional practices. The teachers made various mathematical connections, but differences occurred. The teachers' beliefs about mathematics, especially its interconnected nature, played a role in the mathematical connections each teacher made in practice. Some adult students hold similar beliefs. García-García and Dolores-Flores (2019) found that the pre-university students in their study mostly preferred algorithmic approaches as they believed 'to derive and integrate a function, specific formulas are used' (p. 18). Due to those beliefs, the students established procedural types of connections more often than other types.

The nature of teacher knowledge and beliefs has a long history in the mathematics education research literature. This literature includes Ball *et al.*'s (2008) MKT, in the field of mathematics. A comprehensive review of teacher MKT about functions is found, for example, in Nyikahadzoyi (2015) and Hatisaru (2020). Teacher beliefs related to mathematics and the ways they interact with connection making in teaching practices have been discussed extensively in Singletary (2012) and Askew *et al.* (1997). The existing research, however, has not looked at the ways connections are established in the classroom during teaching, nor the connections within the field of functions, most of the studies being focused on pre-service teachers' (or secondary students') capability in, for example, geometry (e.g., Eli *et al.*, 2013), calculus (e.g., García-García & Dolores-Flores, 2018), sketching the graph of derivative and antiderivative functions (García-García & Dolores-Flores, 2019) or measurement of length (Gamboa *et al.*, 2020). Research on mathematical connections in the classroom has been rare.

2.3. *Purpose of the current study*

Despite attempts by some scholars to develop some sense of what making connections is really like (e.g., Businskas, 2008), comparatively little has been done to describe what this concept looks like especially in the classroom. The present study attempts to identify, and to understand better, the mathematical connections that took place as two secondary mathematics teachers taught functions. Classroom observations data are used to address the following research questions:

(1) What types of mathematical connections do secondary mathematics teachers establish during the teaching of functions, and at what frequency are they observed?
(2) What are the patterns, if any, between teacher profile and connection making?

The following section describes the context for the study (Section 3.1) followed by an overview of the data analysis (Section 3.2). The next section presents the findings based upon the data generated and organised around research questions (Sections 4.1 and 4.2). The final section draws the findings together and interprets them (Section 5).

3. Method: identifying the mathematical connections

3.1. *Context of the study*

Data were obtained as part of Hatisaru's (2014) doctoral research to identify secondary mathematics teachers' MKT about the concept of function and to investigate the associations between teacher MKT and student learning. To achieve the former, a questionnaire was administered to 42 secondary mathematics teachers (31 female and 11 male) to identify their MKT (Hatisaru, 2020). For the latter, case studies of two teachers were carried out. These two teachers were selected from among the 42 teachers. That is, out of 42 teachers 13 volunteered to participate in the second phase of the study, which included classroom observations. Based on their responses to the questionnaire items, the MKT level of two of these teachers was identified as stronger, three teachers as intermediate and eight teachers as weaker. In order to differentiate the influence of MKT on student learning, one teacher from the stronger and one teacher from the weaker group were selected for classroom observations. Interviews and non-participant classroom observations were conducted to capture how the teachers' MKT about the function concept and student learning outcomes interrelated. Both teachers (Fatma and Ali, pseudonyms) were teaching in the same vocational high school located in Ankara (the capital of Turkey). Students were from low- to middle-income families and, overall, their academic performance was poor (for more details on the two case studies, please see Hatisaru & Erbas, 2017).

Ali held a bachelor's degree in mathematics and had received pedagogical training for 4 months before he began teaching. Fatma had a bachelor's degree in mathematics education. At the time of the research, Ali had 14 years and Fatma had 25 years of teaching experience. Based on their responses to the questionnaire measuring secondary mathematics teachers' MKT of the function concept, and a follow-up interview probing their responses and including an additional card sort activity, Fatma's MKT was identified as stronger, whereas Ali's was weaker (Hatisaru & Erbas, 2017). Based on their responses to a semi-structured interview examining five aspects of beliefs about mathematics, including beliefs about mathematics education and teacher knowledge, Ali believed that the goal of mathematics education was to develop students' procedural skills, whereas Fatma believed that the key aim of mathematics education should be to enhance students' logical and critical thinking skills, as well as procedural skills. For Fatma, it was very important for mathematics teachers to have a profound knowledge of mathematics to be able to explain the reasons for facts, rules or procedures (for a full description of the teachers' beliefs, please see Hatisaru, 2018).

Non-participant observations were conducted to collect data—the lessons were observed, listened to, audio recorded and notes were taken (including the whole whiteboard workings) without any intervention. Twenty-four lessons taught by Fatma and Ali (12 lessons per teacher), in Turkish, were observed and all were selected for data analysis to give a comprehensive picture of connections that occurred in their classes. There were 33 students enrolled in Ali's class and 26 students enrolled in Fatma's class. The students in both groups were low achievers in mathematics. Lessons were typically 40 minutes long yielding 16 hours of audio recorded data examined in this study. The classroom observations began on the day the functions unit was first presented and lasted until the primary aspects of functions were no longer the focus of the instruction. In agreement with the curriculum (Turkish Board of Education, 2011, 2018) and associated textbook (Ministry of National Education [MONE], 2012, 2018), the lesson objectives included the following: defining functions and representing them; identifying the domain, codomain and range of a function; determining equality of functions; explaining types of functions; interpreting the behaviour of a function in given intervals; and finding $f + g$, $f—g$, $f \times g$ and f/g where f and g are functions defined from R *to* R.

3.2. *Overview of the data analysis*

Audio recorded data of 24 lessons presented by Fatma and Ali were transcribed and were analysed manually using excel spreadsheets. Data were translated from Turkish to English by the author, who is fluent in both languages. A deductive content analysis was utilised. Qualitative codes were used to generate information about patterns of establishing connections. Occurrences of relational instruction (connections) were used as the unit of analysis, and the established mathematical connections in the teachers' instructions were coded against the analytical framework used in this study (see Figure 1). Frequencies and percentages (rounded to the nearest tenths) were then computed.

The data were coded by me, the author. Before illustrating the coding, I would like to note a related matter. Since mathematics is interconnected, and in general, mathematical connections occur simultaneously (Rodríguez-Nieto *et al.*, 2020), sometimes the type of connection that occurred was open to interpretation. Some of the connections made such as 'linking a table or an expression with the graph' or 'making generalisations based on specific cases' were straightforward and caused no difficulty in interpretation. Some of the connections, however, posed difficulty. For example, the connection 'the

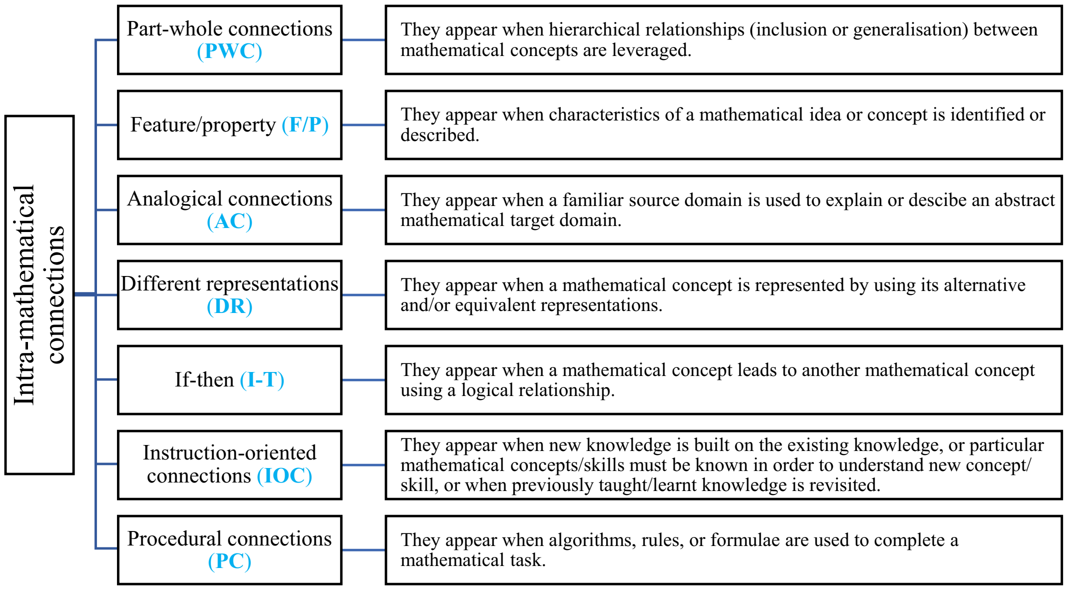


FIG. 1. Analytical framework to study mathematical connections in functions (adapted from García-García & Dolores-Flores, 2019).

range and codomain of onto functions are the same' might be considered I-T or F/P type of connection. That might have weakened the validity of data analysis, and I employed several validation processes to overcome that limitation. Firstly, I discussed the typology employed in this study (Figure 1), along with a subset of data coded by applying this typology, with a colleague who is a competent mathematician and active senior researcher in the field of mathematics education to validate my interpretation. These discussions helped me refine definitions of the categories before applying them to all data. Secondly, when coding the data, I took notes for each occurrence. For example, when a connection was identified as IOC, I noted its reason. Taking notes not only made my thinking explicit, but also allowed me to double-check consistency in the coding. At a third level, Cohen's (1960) Kappa, with 95% confidence interval (CI) estimates (0.86, 0.95), was used to check intra-coder reliability in coding. That is, after I completed the coding of the whole data, I coded these data again to ensure the internal coding consistency of the same coder over 2 months. The Cohen's Kappa for intra-coder reliability was considered satisfactory (0.89) with 95% CI. Finally, I provided a wide range of examples of connections identified in the teachers' instruction, and taken verbatim from lesson transcripts, along with some related diagrams the teachers created or from the textbook (MONE, 2012) (see Section 4.2 and Appendix A). These rich descriptions illuminate some of the important connections in the field of functions produced by participant teachers. They therefore not only contribute to the validity check mechanism but are also useful for understanding the internal connections in functions.

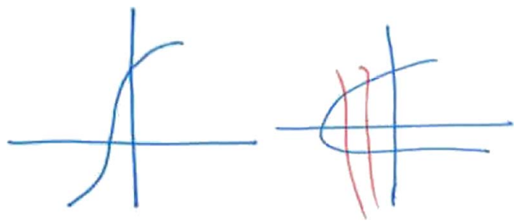
By way of illustration of the coding process, three excerpts are provided below.

Excerpt #1 (Ali, Lesson 3):

. . . Find question 1 in page 77 in your textbook. The *A* contains... [reads the set $\{-5, 1, 0, 2, 3\}$], **the function is defined from *A* to *Z***, that means **the images of *A* elements produced by this function** [outputs] **are integers**. **[I-T] What was *Z*? What would we represent by *Z*?** What was the name of this set? **[IOC: revisiting taught knowledge]** [responds] Integers. . . .

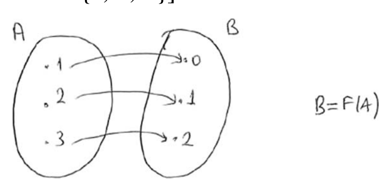
Excerpt #1 hits both I-T and IOC codes as a logical relation is established between the products of function and its range [I-T], and a link is made with previously taught or learnt knowledge [IOC].

Excerpt #2 (Fatma, Lesson 6):
. . . [functions questions are solved] . . . Question 8 in page 108 . . . it is asked which graph is the graph of a function.... Listen! **To decide whether a given graph is a graph of a function or not, we use a test which is called the vertical-line test**. **[PC]** Assume you draw parallel lines to the *y* axis, **like brushing the graph with a comb**, from up to down. **[AC]** The comb touches [cuts] the graph only once. [points at the graph on the whiteboard, below left] Look, here it touches twice. [below right] **EVEN ONCE** [she emphasis that] **it touches the graph more than once, it is not a graph of a function**. **[PWC: generalisation]**. . . . [provides more explanations] [links the test with the uniqueness property] . . . Understand? [the lesson finishes].



Excerpt #2 hits PC, AC, and PWC codes because a procedure (vertical-line test) is used to decide whether the given graph is the graph of a function or not [PC], an analogy is made ('combing the graph') [AC], and the rule (i.e. the graph is not the graph of a function when a drawn vertical line $x = c$ cuts the graph more than once) is generalised [PWC].

Excerpt #3 (Fatma, Lesson 8):... [introduces onto functions by using an analogy]... [writes a function on the whiteboard] $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, $f: A \to B$, $f(x) = x—1$. Let us define whether this function is onto or not. [represents the function in an arrow diagram as shown below **(DR: alternate**)]... Now, **no need to check** this set [***A*—domain of the function**], **it already defines a function**. **[IOC: prerequisite]** [Here, deciding whether the function is an onto or not requires an understanding that it already defines a function] What do we need check? [responds] We check this. [points at *B*–the codomain of the function] Here [in *B*] **there is no leftover elements**. What does that mean? Tell me the codomain of the function, and the range. . . . **No leftover elements in the codomain means, the range and codomain of the function are the same**. . . . [repeats] Please note, **[if] there is no leftover elements in the codomain, the range and codomain are the same**. **[I-T]** [they contain the same elements, in this case $f(A) = B = \{0, 1, 2\}$].



Similarly, Excerpt #3 hits four codes: IOC, AC, I-T and DR. Here, the teacher builds 'onto functions' in students and indicates that the prerequisite of an onto function is satisfying the uniqueness feature

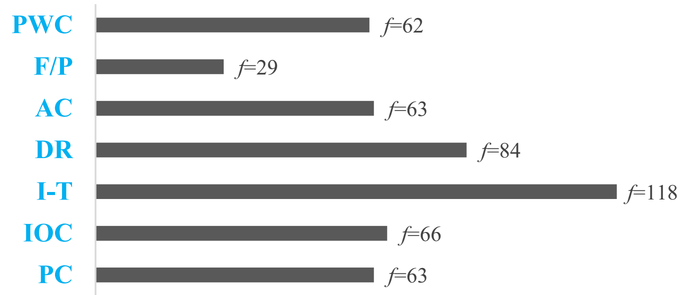


FIG. 2. Frequency *(f)* of mathematical connections identified in the 24 audio recorded lessons.

of functions [IOC]. She uses an analogy linking the 'for every y in the codomain, there is an x in the domain' (target) characteristic of onto functions with 'no leftover elements in the codomain' (source) [AC], and she states that this implies $f(A) = B$ [I-T]. She finally uses an arrow diagram depicting this characteristic of onto functions [DR].

Results are presented in the following section organised by the research questions. The patterns that emerged in the connections made by two teachers across all lessons are given around three connection types: PC, AC and IOC. This is partly for reasons of space, but another good reason is that observed differences in the teachers' connections could be described well around them.

4. Results

4.1. Mathematical connections established by the teachers and their frequencies

Both Fatma and Ali established numerous mathematical connections during the audio recorded lessons. A total of 485 connections were identified in all 24 lessons (12 each). The total number of connections made in each lesson was in the range of four to 33, with a mean of 20.21. Figure 2 captures the frequency distribution of the mathematical connections in seven categories: PWC, F/P, AC, DR, I-T, IOC and PC.

Among them, the I-T type connections were the most made connections ($f = 118$, 24%) followed by the DR ($f = 84$, 17%), with most of them being the use of alternate representations. The frequency of established PWC ($f = 62$, 13%), AC ($f = 63$, 13%), IOC ($f = 66$, 14%) and PC ($f = 63$, 13%) was almost equal. Among all types, the F/P ($f = 29$, 6%) was the least observed type of connection. It is worth noting that while about half of the IOC ($f = 34$) included links to prior knowledge, half of them ($f = 32$) included revisiting taught knowledge or procedures—a new connection type that has emerged from the data in this study.

4.2. Patterns emerged in the type and frequency of connections established

Making mathematical connections was common in both teachers' instructional practices. Nevertheless, differences emerged in the type and frequency of connections made by the teachers, as well as the ways that connections were used in the instruction. A total of 317 units (65%) were identified in Fatma's lessons (mean of 26, range of 8 to 33), and 168 units (35%) were identified in Ali's lessons (mean of 14, range of 4 to 26). As shown in Figure 3, across several types of connections that were coded, Fatma made more connections than Ali, except for PC. She used PWC ($f = 55$, 17%) far more than Ali ($f = 7$,

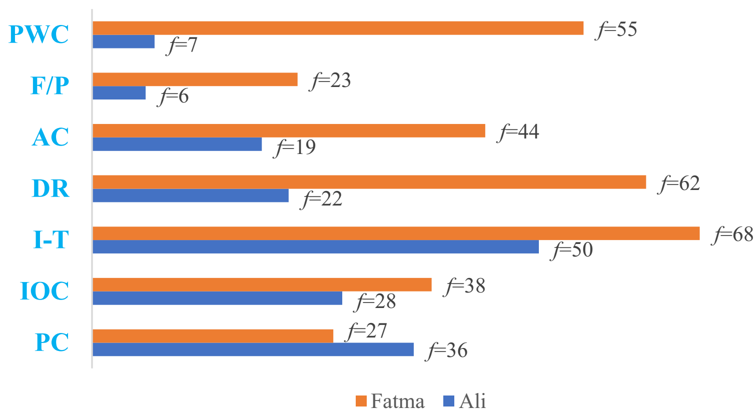


FIG. 3. Frequency *(f)* of mathematical connections identified in the audio recorded lessons of two teachers.

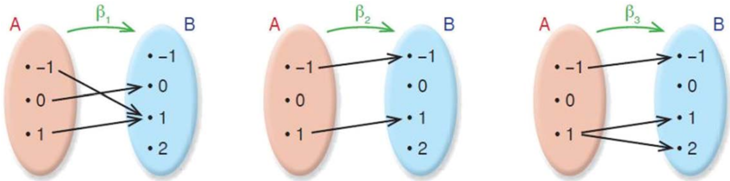
4%), described features of concepts (F/P) ($f = 23$, 7%) almost twice as much as Ali ($f = 6$, 4%) and used DR in her instructions ($f = 62$, 20%) more frequently than Ali ($f = 22$, 13%).

4.3. Patterns emerged in the procedural type connections

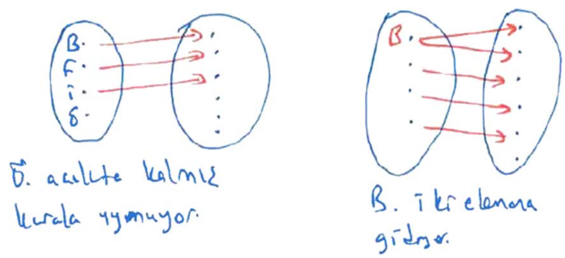
The procedural type connections were less common in Fatma's lessons ($f = 27$, 9%) than Ali's lessons ($f = 36$, 21%). Fatma was less likely to use unlinked procedures or algorithms. When used, she attempted to show the reasoning behind them. For example, the reason behind the vertical-line test, which is used to determine whether a given graph represents a function or not, is the uniqueness feature of functions. As depicted in Excerpt #2 (see Section 3.2), Fatma used a 'comb' analogy to explain the vertical-line test. In this analogy, parallel lines are drawn to the y axis, like brushing the graph with a comb, from up to down (source). The comb touches the graph only once. If it touches the graph more than once, the given graph is not a graph of a function, and this refers to the uniqueness feature of functions (target). Fatma referred to this feature—a function from A to B uniquely associates every element in A with an element in B—not only as the reason for why the vertical-line works but also why not all relations are functions (e.g., Excerpt #4). That is, determining whether the given arrow diagrams are functions requires a connection between the arrow diagram representation of functions and the uniqueness property. As shown in Excerpt #4, Fatma conducted the textbook example by considering this property. To consolidate, she modified the textbook example and created her own additional arrow diagrams. She once more emphasised that because not all elements in the domain correspond to a codomain element, and as one element in the domain corresponds to more than one element in the codomain, neither of these relations defines a function.

Excerpt #4 (Fatma, Lesson 2):
... Now, there is an example in page 74 [shown below]. Let us decide whether they meet our rules or not. [By rules, Fatma means **the uniqueness feature of functions**. Both in Lesson 1 and in this lesson, she refers to this feature: **a function from set *A* to set *B* uniquely associates every element in *A* with an element in *B***] **[F/P]** ... [She runs a class discussion, and they decide that β_1 satisfies

the uniqueness feature whereas β_2, but $\beta 3$ do not meet this feature] . . .



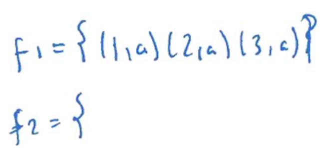
[Summaries] We shall depict these as such [shown below **(DR: alternate)**]. [Here, she modifies the textbook example and creates her own.] In the first diagram, **the element 'Ö' is leftover**—it does not satisfy the rule and **therefore is not a function**. In the second diagram, **the element 'B' has two images**—it does not satisfy the rule. **Also, there are leftover elements in *B*. [I-T]** Understood? . . .



[Ö is leftover, [the relation does] not satisfy the rule; B has two images]

Fatma connected rules or formulae with prior concepts when she used them. In Excerpt #5, for instance, when she solved the relevant question which asks the number of functions defined from *A* to *B* by using the formula, she exemplified some of the functions in sets of ordered pairs and wanted her students write all eight functions defined from *A* to *B* in the same form. She wished her students to *see* that functions are relations that are subsets of $A \times B$.

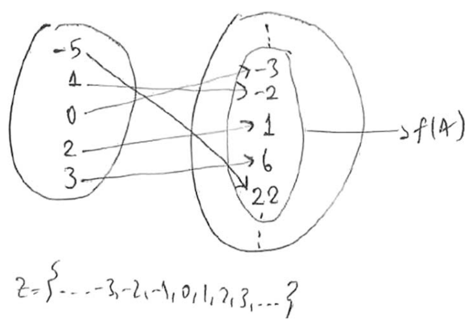
Excerpts #5 (Fatma, Lesson 6):.... [The class practices functions questions in a worksheet. One of them is:] $A = \{1, 2, 3\}$, $B = \{a, b\}$, how many functions can be written from *A* to *B*?... [Through a class discussion Fatma solves the question by first writing f_1 as below **(DR: alternate)**]... It is $\boldsymbol{s(B)^{s(A)} = 2^3 = 8}$, is not it? **[IOC: revisiting taught knowledge]**... This is your homework. **Show all functions and check whether there are eight. [PC].**



Overall, it was evident from the data that Fatma gave attention to helping students to connect procedures they were learning with the concepts that show why the relevant procedures work and/or with the prior or prerequisite concepts.

Explaining the reasons for facts, rules, or procedures was not a common practice for Ali. He often presented only knowledge of routines or procedures and/or presented the content disjointly—as a collection of procedures or formulae. As presented in Excerpt #6, even though alternate representations of the given algebraic function—a set of ordered pairs and an arrow diagram—were created, they were created as standard, discrete routines—the translation between them was unspecified. In other words, no (clear) link was established between the algebraic function and the set of ordered pairs and arrow diagram representations of the same function.

Excerpt #6 (Ali, Lesson 3):... [the class solves a problem in the textbook:] $A = \{-5, 1, 0, 2, 3\}$, $f: A \to Z$, $f(x) = x^2-3$ are given. Represent the function $f: A \to B$ in a set of ordered pairs (p. 77).... [finds the images of set A elements through a class discussion and **represents them in a set** as in the textbook: $\{(-5, 22), (1, -2), (0, -3), (2, 1), (3, 6)\}$] **[DR: alternate]**... Let us represent it [the function] **also in an arrow diagram**. [see below] **[DR: alternate] The set of integers [Z] is a large set**, what I would want you notice is this set [points at the range of the function, i.e., $f(A)$]. This is our range [depicts it in the arrow diagram as below]. **The codomain [Z] contains this set** [$f(A)$]. **[PWC: inclusion]**...



In Excerpt #7, the formula for finding the number of one-to-one functions from A to B was given in addition to a previous formula for finding the number of functions from A to B. No explicit attempt was available to show why these formulae work, or why while the number of functions from e.g., $A = \{a, b\}$ to $B = \{1, 2\}$ is 4, the number of one-to-one functions is 2.

Excerpt #7 (Ali, Lesson 6):... The last time we computed the number of functions defined from set A to set B. [He had used **a formula** for that: **For every non-empty A and B sets, the number of functions defined from A to B is $S(B)^{s(A)}$**] **[PC]** Now, find page 80 in your textbook. In addition to computing the number of functions from A to B, we could also compute the number of one-to-one functions. ... [uses a formula: **Given that $s(A) \geq s(B)$, the number of one-to-one functions defined from A to B is $P(s(A), s(B))$**] **[PC]** ...

4.4. *Patterns emerged in the analogical type of connections*

Teaching practices of the teachers were rich in analogical comparisons. Several AC were identified in the instruction of both Fatma ($f = 44$, 13%) and Ali ($f = 19$, 11%) such as Excerpts #8 and #10 below:

Excerpt #8 (Ali, Lesson 1):
... There is something specific, **like a machine**. It must be something like, assume that there is an abstract object, a machine. For example, **you put the meat in the machine, it is processed and becomes mince** [the input–output relationship] **[AC]** ...

Both of them used analogies, mostly when introducing the concept of function, or the concepts related to functions (e.g., vertical-line test, constant functions), like the participant teachers in

Richland *et al.* (2004). The textbook (MONE, 2012) that they used contain some analogies to (or connections within) functions. Ali did not necessarily use the relevant connections provided by the textbook as elaborated as follows:

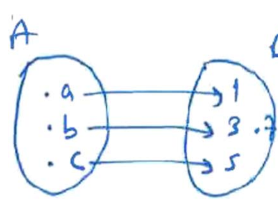
> Excerpts #9 (Ali, Lesson 2):... [presents a question in the textbook in page 79] $A = \{a, b\}$, $B = \{1, 2, 3\}$, and $f: A \rightarrow B$. Find the number of functions from A to B. . . . **How would you find the number of all functions from *A* to *B*?** [Ali provided a formula previously. Here he reminded it.] **[IOC: revisiting taught knowledge]** Listen! If it is from A to B, $s(A) = 2$ and $s(B) = 3$. As it is from A to B, the formula is $s(B)$ to the power of $s(A)$, [$s(B)^{s(A)}$] **[PC]** *3* to the power of *2*—what does that mean? [Students] 9... In the textbook solution of this question, those nine functions were represented in sets of ordered pairs as captured below. This link was not addressed in Ali's instruction; students were only presented the formula.

$f_1 = \{(a, 1), (b, 1)\}$, $f_2 = \{(a, 1), (b, 2)\}$, $f_3 = \{(a, 1), (b, 3)\}$,
$f_4 = \{(a, 2), (b, 1)\}$, $f_5 = \{(a, 2), (b, 2)\}$, $f_6 = \{(a, 2), (b, 3)\}$,
$f_7 = \{(a, 3), (b, 1)\}$, $f_8 = \{(a, 3), (b, 2)\}$, $f_9 = \{(a, 3), (b, 3)\}$
$s(A) = 2$, $s(B) = 3$ ve farklı fonksiyonların sayısı $s(B)^{s(A)} = 3^2 = 9$ dur.

Fatma conducted a similar question, too. She used the set of ordered pairs representation and assigned students to represent the functions in the same form (Excerpts #5).

Fatma produced analogies (and connections) not included in the textbook such as the 'comb' (Excerpt #2) or 'cinema tickets' analogies:

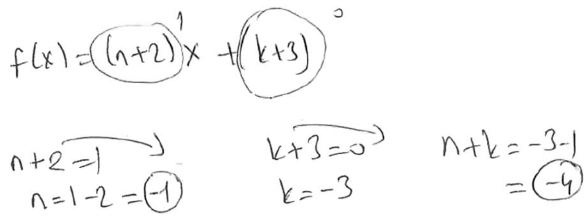
> Excerpt #10 (Fatma, Lesson 6):
> . . . Now, the types of functions. . . . The first type is one-to-one functions. . . . What does [the term] one-to-one mean? . . . **Assume you meet with the school counsellor one on one; what does 'one on one' mean?** [students] One on one. [you and the school counsellor] [Fatma] Right. **The meeting is between you and them—one on one**. **[AC]** As we usually indicate, these terms have specific definitions in mathematics. Normally, **a cinema ticket is for one seat; for instance, Person A has Row K-3; Person B has Row L-11. One ticket is not sold to more than one person**. **[AC]** That is, **each element in *A* corresponds to exactly one element in *B***. [i.e., each x in the domain has exactly one image in the range]... [draws the arrow diagram below **(DR: alternate)**].



4.5. *Patterns emerged in the instruction-oriented type connections*

Both teachers made several instructional type connections. The IOC code was observed in the instruction of Fatma ($f = 38$, 12%) and Ali ($f = 28$, 17%) almost equally. In Ali's case, many of the IOC ($f = 19$) included reminding students of previously taught knowledge or procedures or making them repeat the shown algorithms such as in Excerpts #9 above and #11 as follows:

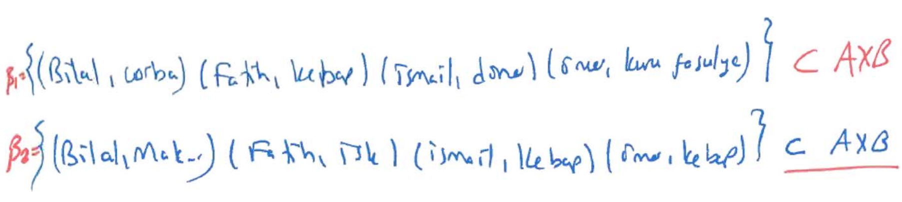
Excerpt #11 (Ali, Lesson 7):...[poses a question] Find $n + k$, given that $f: R \rightarrow R, f(x) = (n + 2).x + (k + 3)$ is an identity function. [Ali] I shall solve this one and will ask you to solve the other two [similar questions]. Listen! Now, **if it was an identity function, was there any other term except for *x*?** [students think] **What was the coefficient of *x*? [IOC: revisiting taught knowledge]** [responds] **If it is an identity function, there is no other terms except for *x*, and the coefficient of *x* is *1*. [I-T]** You will compute this. The procedure for this is, the coefficient of *x* is *1* and the constant is *0*. [see below] **[PC]**...



Fatma too reminded students about learnt or taught knowledge ($f = 15$); yet, she mostly constructed links with prior concepts or knowledge ($f = 23$) (e.g., Excerpt #5), such as with logic, sets, Cartesian product and relations. She presented the content (functions) in an interrelated approach, giving students the chance to learn about their connectedness. According to Fatma, for instance, functions are subsets of the Cartesian product of two sets, *A* and *B*, notated $A \times B$ as evident in Excerpt #12:

Excerpt #12 (Fatma, Lesson 2):... [draws an arrow diagram] [pointing at the domain and range values] These are pairs such as *(1, 2)*. What are these? **They are relations, aren't they?** Yes, **they are relations defined from *A* to *B*, and** [they are] **subsets of *A* × *B*** [cartesian product]. **[IOC: links with prior knowledge]** That is, functions are not totally siloed, they are related [to relations]. But how?...

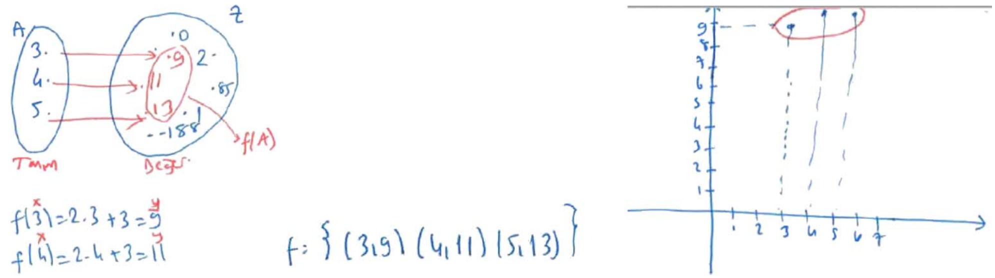
Also, when defining the concept of function in Lesson 1, Fatma defined two sets and wrote some relations. She represented the defined relations (e.g., β_1 and β_2) as follows and notated that they are subsets of $\boldsymbol{A} \times \boldsymbol{B}$:



To Fatma, functions could be represented in pairs and be plotted because they are relations:

Excerpt #13 (Fatma, Lesson 3):

... [defines a function] $A = \{3, 4, 5\}$, $B = \{Z\}$, $f(x) = 2x + 3$ ... **function *f* is, at the same time, a relation [PWC: *inclusion*]**—who is going to write it as relations [pairs]? [conducts a class discussion and represents the function in a set of ordered pairs] ... Nothing was coincidental. After identifying the *x* and *y* values in the arrow diagram, we wrote them as pairs. **This is an arrow diagram representation** [below left]**, this is a set of ordered pairs representation** [below middle], and **we can also show it as a graph**. **[DR: *alternate*]** Who is going to draw its graph? ... [through a class discussion, draws the graph of *f*] [below right].



These differences in the teachers' instruction suggest ways in which the teaching of functions might be improved by building on existing practices.

5. Discussion and conclusions

The types of mathematical connections in two secondary mathematics teachers' instructions on functions were explored, through utilising an extended model for mathematical connections (Rodríguez-Nieto *et al.*, 2020). A large set of classroom observations data (Hatisaru, 2014) was used for the investigation, and all instances of relational instruction—connections—were identified in 24 ninth-grade lessons. Several findings emerged from the data set, which provide insight into emphasising connections in regular mathematics classrooms. I discuss the findings to shed some light on mathematical connections in the classroom and what factors may enable teachers to produce mathematical connections in teaching practices.

It is first evident from the data that the extended model for mathematical connections (Rodríguez-Nieto *et al.*, 2020) is a valid tool to identify the explicit mathematical connections that teachers establish in teaching. The model could be used in future studies to provide the fine-grained mathematical connections in functions and wider content domains. An additional IOC yet unidentified in the literature was evident in participant teachers' teaching: revisiting taught or learnt knowledge. Nevertheless, connecting through frequently revisiting the taught content or procedures might be unique to these two teachers who were teaching vocational high school students whose academic background were comparatively low (e.g., Hatisaru & Erbas, 2013). Teachers of students with stronger academic backgrounds may be less likely to make those connections as frequently, but this is uncertain.

Both Fatma and Ali regularly incorporated mathematical connections into their instruction, like the secondary teachers in Singletary (2012) or the prospective middle teachers in Eli *et al.* (2013). Making mathematical connections was common in their instructional practices independent of their MKT level (Eli *et al.*, 2013) or beliefs (Singletary, 2012). However, differences occurred both in terms of the frequency of connections made by them (65% and 35%, respectively) and their types. Fatma established PWC, DR and F/P type of connections far more frequently than Ali, and Ali made more PC. The study suggests that this 'connections gap' may reflect the differences in the teachers' knowledge (e.g., Askew *et al.*, 1997; Hughes, 2016) and/or in their beliefs about the nature of mathematics and teaching and learning of it (e.g., Singletary, 2012). That is, what distinguished Fatma from Ali was a sound understanding of the concept of function (Hatisaru & Erbas, 2017) and a particular set of coherent beliefs about mathematics, the goals of mathematics education and the teaching and learning of mathematics (Hatisaru, 2018), which might underpin her teaching of functions. It seemed that Fatma viewed mathematics as a connected discipline and aimed for her students to acquire knowledge of functions based on an integrated network of understanding. It is necessary to deepen the study of relationships between making mathematical connections, content knowledge and beliefs as in this study we only know Fatma's content knowledge about the concept of function and her epistemic beliefs not specifically on making mathematical connections.

Ali had a mathematics degree, and four months of pedagogical training before starting his teaching career, whereas Fatma had a degree in mathematics education. Ali displayed teaching that was relatively compartmentalised and, in most cases, his teaching was reliant on procedures, rules, formulae and algorithms, whereas Fatma established connections among different mathematics content domains and different ideas in the same content domain of mathematics, through using a variety of explanations, diagrams, graphs, examples and counter examples, reasoning and generalisations. The study shows that having a degree in mathematics may not necessarily associate with having an integrated network of understanding or a rich network of connections between different mathematical ideas. On the other hand, we know neither Ali's nor Fatma's knowledge of advanced mathematics was obtained from their university education programs. What mathematical connections are available in mathematics and/or mathematics education at university or college level programs, and as Zazkis & Mamolo (2011) suggested, 'what teachers' knowledge "beyond school curriculum" can bring to teaching' (p. 9) need to be further explored.

The question arising from the data in this study is whether textbooks have influence on teachers in making mathematical connections. That is, the textbook (MONE, 2012) that Fatma and Ali used contains some mathematical connections in functions. Ali adhered to the textbook. Nevertheless, while he followed the textbook scripts with little to no deviation, he did not necessarily use the relevant mathematical connections provided by the textbook. Fatma was less dependent on the textbook. When she used it, she modified the explanations, activities or examples to her own teaching style and created her own examples or representations. She produced numerous connections not included in the textbook. This is where the issues are complex and several factors may be in play. On a relevant and important note, the present study focused only on teacher-produced mathematical connections. Researchers in this field are encouraged to examine mathematical connections available in textbooks, and their conceptual strengths, by using the model utilised in this study.

The role of connection making in learning school mathematics with understanding has been widely supported in the field of mathematics education (e.g., Bossé, 2003; Cai & Ding, 2017). As mathematical concepts cannot be understood in isolation, the mathematical connections constructed in the classroom give opportunities for students to relate mathematical ideas, concepts, meanings and procedures to each other, which aid in mathematical understanding (Singletary, 2012; Gamboa *et al.*, 2020), and to see mathematics as an integrated whole (Evitts, 2004). The differences between the mathematical connections Fatma and Ali made suggest that the students in each of their classes had different opportunities to learn mathematics. Future research is needed to understand how student learning is influenced by the connections experienced in the classroom.

Finally, school curricula internationally (e.g., ACARA, 2018: NCTM, 2000, 2014; MOE, 2012; MOE, 2018; Turkish Board of Education, 2011, 2018) call for mathematical connections, both in mathematics and between mathematics and other subject areas. However, internal connections within or between mathematical topics have sometimes been backgrounded (Bossé, 2003) and/or not documented widely. Contributing to the literature, this study documents some of the important mathematical connections in functions through comprehensive descriptions of episodes from two secondary mathematics teachers' lessons (Sections 3.2 and 4.2; Appendix A). Documenting them is significant because they inform our understanding of the internal connections within functions. In addition, they can be emphasised in mathematics instruction and learning. Sometimes determining the type of a particular mathematical connection may be difficult. For example, the connection 'the range and codomain of onto functions are the same' may be considered either I-T or F/P type of connection, or both. And this again shows the interconnected nature of mathematics.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Disclosure statement

No potential conflict of interest was reported by the author.

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Appendix A: Sample mathematical connections established by the teachers

Excerpt #14 (Fatma, Lesson 2):

> ... [defines the concept of function]... They [functions] are notated in different. Which ones do we mostly use? We mostly use this [points at the whiteboard, $f(x) = y$], and we use this [$f\colon A \to B$] when we define a function. When we state $\boldsymbol{f(x) = y}$, x elements are in set A and y elements are in set B. Note that $\boldsymbol{y}$ **is the same as** $\boldsymbol{f(x)}$. **[DR: *equivalent*]** What we say is, y is the same as $f(x)$....

Excerpt #15 (Fatma, Lesson 10):

> . . . Let us define a function such as $f(x) = 2x + 1$ **from *R* to *R***. [Real numbers] This means that ***x* can be any number**—it can be 0, 1, 2, 89, or 876. **Any number**. What meant by 'any number' is, **for any *x*, you find exactly one *y* value** corresponding it. **[I-T].**

Excerpt #16 (Fatma, Lesson 2):

> ... **Functions are relations**. You have been already introduced to relations, but why you are now introduced to functions as a separate subject? Remember **relations were in the form of pairs, i.e. *(x, y)***. But why we are now back to relations?... [responds] They are not the same. But what is the difference? We will now learn how they are different. In fact, **all functions are relations**, but they have particular conditions. **Not all relations are functions**. **[PWC: inclusion]**... [describes the uniqueness feature of functions (see Excerpt #4)].

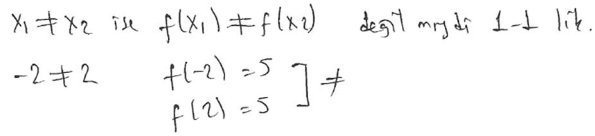
Excerpt #17 (Fatma, Lesson 8):

> In page 82 in your textbook, the first question. [writes it on the whiteboard] $f\colon Z \to R$, $f(x) = 2x + 1$. From Z to R, read carefully, from integers to real numbers. It is asked whether the function is an onto function.... [students] Let us guess and check. [Fatma] No, **sometimes we cannot guess and check** [according to Fatma, then the process would be cumbersome]. There must be a way. **If** $\boldsymbol{y = 2x + 1}$**, then** $\boldsymbol{x = (y—1)/2}$. Now, think, **as every *y* are real numbers, every *x* you find are integers**. **[PWC: generalisation]**...

Excerpt #18 (Ali, Lesson 6):

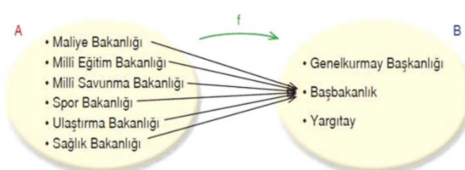
> . . . [poses a question] $A = \{-2, -1, 0, 1, 2\}$, $B = \{1, 2, 5\}$, $f\colon A \to B$ and $f(x) = x^2 + 1$. Find the range of the function and show it in an arrow diagram. . . . [after solving the question through a class discussion, asks:] Is this function one-to-one? [students] One-to-one. [Ali surprises] The images of

both -2 and $+2$ are the same. [i.e. *5*] Are the pre-images [-2 and $+2$] same? . . . [only a few students notice that] [He reminds one-to-one functions **(IOC: revisiting taught knowledge)** and writes the symbolic statements shown below on the whiteboard **(DR: alternate)**] . . . It is not one-to-one. **For one-to-one functions, no two elements in the domain of *f* map to the same element in the range of *f*. That is, no *y* in the range is the image of more than one *x* in the domain**. **[PWC: generalisation]** [in this case, *5* is the image of both -2 and $+2$ which violates this condition] . . .



Excerpt #19 (Ali, Lesson 8):

... Listen! There is an arrow diagram in page 83 in the textbook [shown below], will you view it? [Students: yes] [This diagram shows that **all elements in *A*—Ministries—are linked to exactly one element in *B*—the Prime Ministry**] **[AC] Where are all Ministries linked to?** [Students: to the Prime Ministry] [Ali] In other words, **what is the image of each Ministry produced by this function**? [responds] The Prime Ministry. They [each element in *A*] correspond to a CONSTANT [emphasis that] in *B*. Is there any other element in *B* that they correspond to? [responds] No. Assume **a function that the image of the elements in *A* is independent of itself—all elements in *A* have an exact image, a constant**. **[F/P]** What this is called is constant function.



[A: Ministries of Finance, Education, Defence, Sport, Transportation, Health; B: Presidency of General Staff, Prime Ministry, Supreme Court]

Excerpt #20 (Ali, Lesson 7):

. . . Please note the definition of the identity function in page 82 in the textbook. When you note, notice that the image of *1* is *1*, *2* is *2*, *3* is *3*, and so on. What we call **an identity function is, the image of a pre-image is itself** [i.e. *(1, 1), (2, 2), (3, 3)*], and we **represent it as $I(x) = x$**. **[DR: alternate]** You only find the term *x* in identity functions, not e.g., x^2 or any constant. **The coefficient of *x* in $I(x) = x$ is *1***. **[F/P]**...

Excerpt #21 (Ali, Lesson 6):

... There are also into functions, let us define them, too.... **An into function is not an onto function**, that is, $f(A) \neq B$ [for onto functions $f(A) = B$] **[IOC: links with prior knowledge]** What does that mean? The range [*f(A)*] of an into function is not equal to codomain [*B*] of the function.... [gives examples such as $B = \{1, 2, 3, 4, 5\}$ and $f(A) = \{1, 2, 3\}$].

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