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Constructivist Learning Method of Ordinary Differential Equations in College Mathematics Teaching

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Abstract

The article uses ordinary differential to solve inhomogeneous equations by the constructivist learning concept. We use the equivalent equations to study the n-th order non-homogeneous linear ordinary differential equations with constant coefficients and get the method of solving this equation. Then we use the Filippov transformation and comparison theorem to prove the boundedness of all system trajectories. Numerical results show that the calculation formula is effective for solving stiff ordinary differential equations.

Keywords: Ordinary differential equations; Constructivism; College mathematics teaching; Riccati method
AMS 2010 codes: 34A25

1. Introduction

Ordinary differential equations' two-end boundary value problem (TPBVP) is frequently encountered in scientific engineering calculations. The classic solution methods include traditional methods such as the "shooting method," finite difference method, and finite element method [1]. These methods are often unsatisfactory in terms of calculation accuracy and stability. So we first consider the following initial value problem:

$$\dot{v} = Hv + r(t); v(0) = v_0 \tag{1}$$

H is the $2n \times 2n$ dimensional stationary system matrix. $r(t)$ is an arbitrarily varying non-homogeneous term v_0 is a given initial condition [2]. The differential equation (1) is a first-order linear non-homogeneous ordinary differential equation about the $2n$ dimensional state vector v . If the non-homogeneous term $r(t) = 0$, then the differential equation (1) degenerates into a first-order linear homogeneous ordinary differential equation. Linear

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means that the system matrix H and the inhomogeneous term $r(t)$ have nothing to do with v, while first-order means that the differential equation has only the first derivative concerning v. The corresponding first-order linear non-homogeneous boundary value problem can be described as

$$\dot{v} = Hv + r(t); q(0) = q_0, q(t_f) = q_{t_f} \tag{2}$$

$v = (q^T \quad p^T), t_f$ is the length of the solution area. q_0 and q_{t_f} are the given boundary conditions at both ends.

Therefore, the problem addressed in this article is the first-order linear non-homogeneous two-end boundary value ordinary differential equation described in equation (2). The solution of the differential equation (1) can be expressed as

$$v(t) = e^{Ht} v_0 + \int_0^t e^{H(t-\tau)} r(\tau) d\tau \tag{3}$$

In the above formula, the first term at the right end is the general solution of the homogeneous equation corresponding to equation (1), and the second term at the right end is the special solution of the equation (1) corresponding to the non-homogeneous equation. In numerical calculation, we discretize the time domain into a series of equal intervals with a step length of η. Then at any time $t_k = k\ \eta (k = 0,1,2,\cdots)$ can get the recurrence relationship of the state vector v in a time step by formula (3):

$$v_{k+1} = \Phi_0(\eta) v_k + \int_0^\eta e^{H\tau} r(t_{k+1} - \tau) d\tau; \Phi_0(\eta) = e^{H\eta} \tag{4}$$

If the missing n initial states can be obtained by using the n states in the terminal boundary of the boundary value problem at both ends, then the boundary value problem at both ends will be transformed into an initial value problem. In this way, the recursive formula (4) can obtain the state vector v_k at any discrete-time t_k. The following gives the precise integration calculation process of the homogeneous term state transition matrix $\Phi_0(\eta)$ and the non-homogeneous term $\int_0^\eta e^{H\tau} r(t_{k+1} - \tau) d\tau$.

2. Precise integration of the homogeneous general solution $\Phi_0(\eta)$

The precise integration of the general solution $\Phi_0(\eta)$ of the homogeneous equation can be calculated using the matrix index method or the section mixed energy method. The matrix index method is simple in principle and easy to program [3]. The zone-mixed energy method is very suitable for solving various problems in modern (optimal) control theory. The following two methods are used to calculate the state transition matrix $\Phi_0(\eta)$ of the homogeneous general solution.

2.1 Matrix index method

Assume that the area to be solved for the entire boundary value problem at both ends is $[x_0, x_f]$, and the length is $L = x_f - x_0$. We discretize L into a series of segments at equal distances, then the segment length is η:

$$\eta = L / 2^M \tag{5}$$

Among them M is a positive integer, which controls the size of the discrete step. In the basic section η, we use the fine integration method to calculate the transfer matrix $\Phi_0(\eta)$ of the section

$$\Phi_0(\eta) = e^{H\eta} = (e^{H\eta/2^N})^{2^N} \tag{6}$$

In this way, the state transition matrix $\Phi_0(\eta)$ with the section length η is obtained by the standard fine integration method [4]. Further, execute the class 2^N algorithm to obtain the state transfer matrix $\Phi(L)$ of the entire section:

$$\Phi(L) = e^{HL} = (\Phi_0(\eta))^{2^M} \tag{7}$$

At this time, the terminal state vector can be expressed as the initial state vector as

$$\begin{pmatrix} q_{xf} \\ p_{xf} \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} q_{x_0} \\ p_{x_0} \end{pmatrix} \tag{8}$$

$$\Phi(L) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$$

We use equation (8) to connect the initial state and the end state through the state transition matrix $\Phi(L)$ of the entire section.

2.2 Sectional mixed energy method

By establishing the connection between the state differential equation coefficient matrix and the mixed energy matrix and then solving the differential equation that the mixed energy matrix satisfies, the solution of the asymmetric Riccati differential equation at each time point is obtained [5]. This method can also be used to solve the state transition matrix $\Phi_0(\eta)$. The homogeneous differential equation of state is

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & D \\ B & C \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{9}$$

q is the n dimensional state vector. p is the m dimensional state vector. A, D, B, C are the coefficient matrix with the corresponding dimension. We use the section mixed energy method to describe. The state at both ends of the section is described by the mixed energy matrix as

$$\begin{pmatrix} q_b \\ p_a \end{pmatrix} = \begin{pmatrix} F & -G \\ Q & E \end{pmatrix} \begin{pmatrix} q_a \\ p_b \end{pmatrix} \tag{10}$$

q_a, p_a are the state vector at the a end of the section. q_b, p_b is the state vector at the b end of the section. F, G, Q, E are the mixed energy matrix of this section. The section mixing energy matrix also has section merging formulas [6]. Assume that section $1:[x_a, x_b]$ and section $2:[x_b, x_c]$ are as shown in Figure 1. The segment mixing energy matrix is Q_1, G_1, F_1, E_1 and Q_2, G_2, F_2, E_2. We combine the two sections into a whole. The zone mixing energy matrix Q_c, G_c, F_c, E_c corresponding to zone $3[x_a, x_c]$ is shown in the following formula.

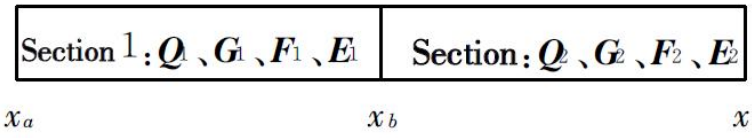

Figure 1 Schematic diagram of segmental mixed energy merging

Sections 1 and 2 are merged into the whole section 3, and the section mixed energy merge formula:

$$\begin{aligned} Q_c &= Q_1 + E_1(I_m + Q_2G_1)^{-1}Q_2F_1 \\ G_c &= G_2 + F_2(I_n + G_1Q_2)^{-1}G_1E_2 \\ F_c &= F_2(I_n + G_1Q_2)^{-1}F_1 \\ E_c &= E_1(I_m + Q_2G_1)^{-1}E_2 \end{aligned} \tag{11}$$

We use type 2^N algorithms in the section of length η. The article divides the section of length η into smaller sections of length $\tau = \eta / 2^N$. At the same time, we expand the segment mixing matrix series in this very small segment τ. According to the equation satisfied by the section mixing energy matrix, the connection between the section mixing energy matrix and the coefficient matrix of the homogeneous state differential equation (9) is established to obtain the analytical expression of the section mixing energy matrix in a very small section τ. The mixed energy matrix $Q(\eta), G(\eta), F(\eta), E(\eta)$ with the section length η is obtained by the section merging formula. Then execute the mixed energy combination formula (11) of the whole section L to obtain the mixed energy matrix $Q(L), G(L), F(L), E(L)$ of the whole section [7]. Therefore, the two states are connected, as shown in equation (10). We express it in the form of (8) transfer matrix

$$\begin{aligned} \Phi_{11} &= F(L) + G(L)E^{-1}(L)Q(L) \\ \Phi_{12} &= -G(L)E^{-1}(L) \\ \Phi_{21} &= -E^{-1}(L)Q(L) \\ \Phi_{22} &= -E^{-1}(L) \end{aligned} \tag{12}$$

Equation (12) is the corresponding relationship between the state transition matrix $\Phi(L)$ and the mixed energy matrix $Q(L), G(L), F(L), E(L)$ in the section L.

3. Fine integration of inhomogeneous terms

We use the 2^N type algorithm to expand the approximate series of the inhomogeneous terms in a very small section τ. Then use the method of section merging to obtain the expression of the inhomogeneous terms in the introductory section η. The following gives specific derivations for inhomogeneous terms as polynomials, trigonometric functions, and exponential functions [8]. If the non-homogeneous terms are in polynomial form, for example, the following linear non-homogeneous terms

$r(x) = r_k + r_1 (x - x_k); r_1 = (r_{k+1} - r_k) / \eta$

Then the corresponding inhomogeneous term in equation (4) is specifically solved as

$F(\eta) = \Phi_1(\eta) r_{k+1} - \Phi_2(\eta)(r_{k+1} - r_k) / \eta$

In

$\Phi_1(\eta) = \int_0^\eta e^{H\tau} d\tau; \Phi_2(\eta) = \int_0^\eta e^{H\tau} \tau d\tau$

After the partial integration operation, $\Phi_1(\eta), \Phi_2(\eta)$ itself has the section merging formula expressed as follows

$$\begin{aligned} \Phi_1(2\eta) &= \Phi_1(\eta) + \Phi_0(\eta)\Phi_1(\eta); \\ \Phi_2(\eta) &= \Phi_2(\eta) + \Phi_0(\eta)(\eta\Phi_1(\eta) + \Phi_2(\eta)) \end{aligned} \tag{13}$$

The section matrix $\Phi_1(L), \Phi_2(L)$ of the entire section L is obtained by executing the 2^N typing algorithm. This process is expressed in programming language as

$for(i=0;i<M;i++)$

{

$\Phi_1=\Phi_1+\Phi_0\Phi_1;$

$\Phi_2=\Phi_2+\Phi_0(\eta\Phi_1+\Phi_2);$

$\eta=\eta+\eta;$

}

After we get $\Phi_1(L),\Phi_2(L)$, the inhomogeneous term of the entire section L can be expressed as

$$F(L)=\Phi_1(L)r_{x_f}-\Phi_2(L)(r_{x_f}-r_{x_0})/L$$

If the inhomogeneous term is a sine/cosine function, there is an expression

$$r(x)=r_1\sin(\gamma x)+r_2\cos(\gamma x)$$

Then the special solution corresponding to the inhomogeneous term is

$$F(\eta)=E_c(\eta)a_{k+1}+E_s(\eta)b_{k+1}$$

In

$$E_c(\eta)=\int_0^\eta\cos(\gamma\tau)e^{H\tau}d\tau$$

$$E_s(\eta)=\int_0^\eta\sin(\gamma\tau)e^{H\tau}d\tau$$

$$a_{k+1}=r_1\sin(\gamma x_{k+1})+r_2\cos(\gamma x_{k+1})$$

$$b_{k+1}=-r_1\cos(\gamma x_{k+1})+r_2\sin(\gamma x_{k+1})$$

Among them, $E_c(\eta),E_s(\eta)$ can also be calculated by fine integration. Similarly, there is the following section merging formulas through the partial integration operation:

$$\begin{aligned}E_s(2\eta)&=E_s(\eta)+\Phi_0(\eta)(\cos(\gamma\eta)E_s(\eta)+\sin(\gamma\eta)E_c(\eta))\\E_c(2\eta)&=E_c(\eta)+\Phi_0(\eta)(\cos(\gamma\eta)E_c(\eta)-\sin(\gamma\eta)E_s(\eta))\end{aligned}\tag{14}$$

We use the 2^N typing algorithm to get the $E_c(L)$, $E_s(L)$ of the entire section L, which is expressed in programming language as

$for(i=0;i<M;i++)$

{

$E_s=E_s+\Phi_0(\cos(\gamma\eta)E_s+\sin(\gamma\eta)(\gamma\eta)E_c);$

$E_c=E_c+\Phi_0(\cos(\gamma\eta)E_c+\sin(\gamma\eta)(\gamma\eta)E_s);$

$\Phi_0=\Phi_0\ \Phi_0;$

$\eta=\eta+\eta;$

}

After obtaining the matrix $E_c(L),E_s(L)$, the inhomogeneous term of the entire section L can be expressed as

$$F(L)=E_c(L)a_{x_f}+E_s(L)B_{x_f}$$

If the inhomogeneous term is an exponential function, the expression is as follows

$$r(x)=e^{ax}\quad f$$

Then the special solution corresponding to the inhomogeneous term is

$$F(\eta)=e^{ax_{k+1}}\quad E_e(\eta)\quad f$$

In

$E_e(\eta) = \int_0^{\eta} e^{H\tau} e^{-a\tau} d\tau$

After the division integration calculation, the following section merging formula is available:

$E_e(2\eta) = E_e(\eta) + e^{-a\eta} \Phi_0(\eta) E_e(\eta)$

The $E_e(L)$, $E_s(L)$ of the entire section, L is obtained by executing the typing 2^N algorithm, which is expressed in programming language as

$for(i = 0; i < M; i++)$

{

$E_e = E_e + e^{-a\eta} \Phi_0 E_e;$

$\Phi_0 = \Phi_0 \quad \Phi_0;$

$\eta = \eta + \eta;$

After executing the above program statement, the inhomogeneous term expression of the entire section L is obtained:

$F(L) = e^{ax_f} \quad E_e(L) \quad f$

When the inhomogeneous term is the above function itself or a combination between them, the numerical result calculated by fine integration is almost the exact solution on the computer. When the inhomogeneous terms are other functions, it can be expressed in section η by fitting polynomials, Fourier series expansion, and other methods into the above functional form [9]. At this time, the calculation accuracy of the inhomogeneous term mainly depends on the accuracy of this approximation.

4. Precise integration solution for the boundary value problem at both ends of the linear non-homogeneous equation

With the above precise integration process of the homogeneous term state transition matrix $\Phi(L)$ and the non-homogeneous term $F(L)$, we can consider the precise integration method for the boundary value problem at both ends of the linear non-homogeneous equation [10]. The following equation can connect the boundary value problem of the state vector at both ends

$$\begin{pmatrix} q_{x_f} \\ p_{x_f} \end{pmatrix} = \Phi(L) \begin{pmatrix} q_{x_0} \\ p_{x_0} \end{pmatrix} + F(L) \tag{15}$$

In

$\Phi(L) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, F(L) \begin{pmatrix} F_q \\ F_p \end{pmatrix}$

If the boundary conditions at both ends are known, such as q_{x_0} and q_{x_f}, the state vector p_{x_0} with an unknown initial end can be obtained:

$p_{x_0} = \Phi_{12}^{-1}(q_{x_f} - \Phi_{11} q_{x_0} - F_q)$

In this way, the complete initial conditions q_{x_0} and p_{x_0} are obtained. At this time, it can be solved completely according to the initial value problem [11]. When the sections are merged, the state transition matrix $\Phi_0(\eta)$ and the inhomogeneous term $F(\eta)$ of section η can be stored. In this way, the state vector value vk at the endpoint k of each section can be solved according to the precise integration recursive format of the initial value problem.

5. Numerical examples

Solve the boundary value problem of a second-order ordinary differential equation $y''(x)+\frac{1}{4}y(x)=8$. The boundary condition at both ends is $y(0)=0, y(10)=0$. The analytical solution is

$$y(x)=32\left(\frac{\cos 5-2}{\sin 5}\sin\frac{x}{2}-\cos\frac{x}{2}+1\right)$$

Select $M=5$ as the step parameter. The absolute error accuracy e calculated by the method in this paper is shown in Figure 2. It can be seen from Fig. 2 that the absolute error accuracy of the calculation result is in $10^{-13}-10^{-16}$ regardless of whether the matrix index method or the Riccati method is used. Both have quite high calculation accuracy and almost reach the exact solution on the computer.

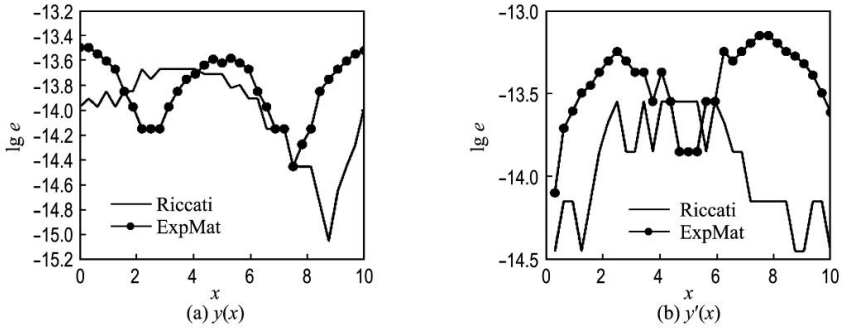

Figure 2 Example 1 The matrix index method and Riccati method calculate the absolute error of the function value and its first derivative

Example 2: We solve a boundary value problem at both ends of a stiff ordinary differential equation. The differential equation is

$$\begin{pmatrix} du/dx \\ d\upsilon/dx \end{pmatrix}=\begin{pmatrix} 998 & 1998 \\ -999 & -1998 \end{pmatrix}\begin{pmatrix} u \\ \upsilon \end{pmatrix}+\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1+x)e^{-x}$$

The eigenvalues of the system matrix H are $\lambda_1=-1$, $\lambda_2=-1000$ respectively, and the difference is relatively large. It has obvious rigid characteristics. The given boundary conditions are

$$u(0)=1, \upsilon(1)=\frac{331\,669}{332\,667}e^{-1000}-\frac{1328\,671}{332\,667}e^{-1}$$

At this time, the analytical solution is

$$u^*(x)=-\frac{331\,669}{332\,667}e^{-1000x}+\frac{664\,336}{332\,667}e^{-x}+\frac{1331}{333}e^{-x}x+2e^{-x}x^2$$

$$\upsilon^*(x)=-\frac{331\,669}{332\,667}e^{-1000x}-\frac{331\,669}{332\,667}e^{-x}-\frac{665}{333}e^{-x}x-e^{-x}x^2$$

For this rigid problem, we use the method of this paper to calculate. Select $M=5$ for the long parameter, and calculate the absolute error as shown in Figure 3. It can be seen from the figure that the calculation accuracy of this method is also in the $10^{-10}-10^{-16}$ order. The Riccati method has a slightly lower calculation accuracy than the matrix index method. Because in the process of calculating the zone mixing energy matrix and then transforming it into the state transition matrix, it is necessary to invert part of the matrix. This will cause numerical errors in the inversion process for severe problems.

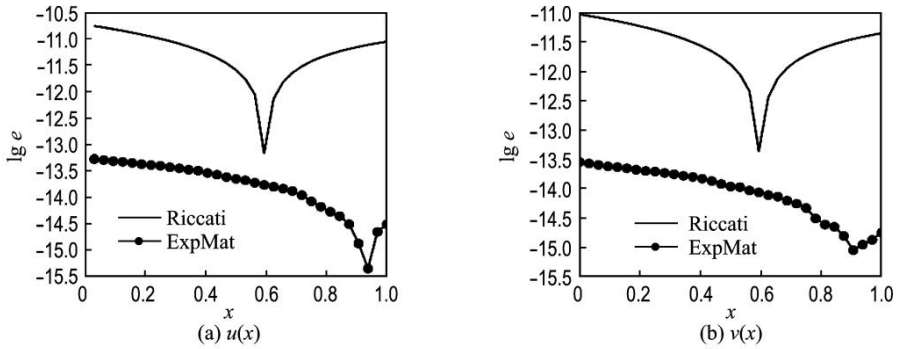

Figure 3 Calculation example 2 The absolute error of the matrix index method and the Riccati method to calculate the function value

6. Conclusion

This paper adopts the precise integration method to calculate the state transition matrix and non-homogeneous term of the homogeneous general solution. At the same time, we transformed the boundary value problem at both ends of the linear non-homogeneous ordinary differential equation into the initial value problem. We introduced the recursive form of the precise integration solution process. The method in this paper only needs to perform a precise integration operation on the system matrix once. The article does not need to invert the system matrix because there is no approximation of the different classification algorithms for inhomogeneous terms. Numerical examples show that the algorithm has high reliability and accuracy.

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