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Developing an Understanding of what Constitutes Mathematics Teacher Educator PCK: A Case Study of a Collaboration Between Two Teacher Educators

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Abstract: Previous research into the knowledge required for teaching has focused primarily on pre-service and in-service teachers' knowledge. What is less researched, however, is the role of the teacher educator in helping pre-service teachers (PSTs) develop the knowledge needed in order to teach mathematics to students. The focus thus shifts from examining school teachers' knowledge for teaching mathematics to school students, to studying teacher educators' knowledge for teaching teachers. This raises the question of what is the nature of this knowledge as required by teacher educators, and how evident is it in their practice? This paper documents the interactions among two teacher educators and two cohorts of PSTs enrolled in a unit designed to teach mathematics pedagogy to early childhood and primary PSTs. Over one semester, two teacher educators observed each other's classes, engaged in reflective professional conversations, and surveyed PSTs about lesson material and delivery. The results indicated there were a number of issues faced by the teacher educators that could be interpreted through the use of a teacher knowledge framework, with examples for this study focussing on a representative lesson. The findings add to the field of research into teacher educator knowledge and have implications for mathematics teacher educators and the pre-service teachers they teach.

Introduction

Shulman's (1987) identification of the different knowledge types required for teaching has resulted in a body of literature that has particularly examined teacher and pre-service teacher (PST) knowledge, especially in relation to their Pedagogical Content Knowledge (PCK). Shulman (1987, p. 8) described PCK as:

the blending of content and pedagogy into an understanding of how topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category [of teacher knowledge] most likely to distinguish the understanding of the content specialist from that of the pedagogue.

In terms of mathematics teaching, PCK is needed for teaching different mathematical topics, in order to make these topics comprehensible to learners. This knowledge is also central to understanding student misconceptions; knowing how topics are organised and

taught; having a repertoire of representations, explanations, and examples that illustrate concepts; and having the ability to adjust lessons to cater for all learners (Shulman, 1986). Here the purpose of PCK is to help students to learn mathematics. However, where and how do teachers learn this PCK? Just as the challenge for mathematics teachers is to determine how to teach their students mathematics in an effective way, the challenge faced by teacher educators lies in determining how to teach PSTs the PCK that is needed to teach their future students. Consequently, what PCK do teacher educators need to draw upon for teaching PCK? This is an important question for the development of teacher educators in a professional capacity. While PSTs are prepared for their role as teachers through a combination of formal training and practicum placement, teacher educators' prior experience may predominantly be in the teaching of children. What skills and knowledge for teaching PSTs are additional to this role, and how do we begin to identify them? While investigation into in-service and pre-service teacher PCK has been extensively researched (e.g., Chick, Pham & Baker, 2006; Hill, Ball & Schilling, 2008; Maher & Muir, 2013), there has been less research into the knowledge required by mathematics teacher educators (MTE), the "knowledge for teaching knowledge for teaching mathematics" (Beswick & Chapman, 2012, p. 2).

Beswick and Chapman (2012) identified this area as an emerging field of research, and attested that the articulation of this knowledge could prove beneficial to prospective and practicing teacher educators, institutions who educate or hire teacher educators, and prospective and practicing teachers. With this in mind, the research in this paper examines the knowledge that teacher educators seem to bring into action when working with PSTs. In particular, it looks at what might be called "teacher educator PCK" and addresses the following research questions:

- What aspects of MTE PCK are evident when teacher educators observe each other's practice?
- In what ways do MTEs make their teaching strategies explicit for PSTs?
- Are the components of the "Knowledge Quartet" framework, previously used for examining mathematics teacher PCK, appropriate to use in the interpreting of MTEs' PCK practices?

Review of Literature

Knowledge for Teaching

Research into the different types of knowledge required for teaching has been well documented (e.g., Chick, et al., 2006; Hill, et al., 2008; Ma, 1999; Rowland, Turner, Thwaites & Huckstep, 2009; Shulman, 1986). Shulman's (1987) theoretical framework described seven categories of teacher knowledge, which became the foundation for describing the knowledge base for teaching. His conceptualisation of PCK has become a particular focus for recent mathematics education research.

Since the conceptualisation of this knowledge, a number of researchers in mathematics education have developed their own frameworks for understanding the different types of knowledge required for teaching (e.g., Chick et al., 2006; Rowland et al., 2009). These frameworks have some features in common, and identify aspects of PCK for teaching school mathematics. Chick et al. (2006), for example, developed a framework that identified a range of component knowledge areas in three different categories: "clearly PCK", "content knowledge used in a pedagogical context", and "pedagogical knowledge used in a content context". The framework can, and has been, used to interpret teachers' PCK for teaching school mathematics through evidence of practices such as their use of examples,

representations of concepts, and curriculum knowledge. Practical applications of the framework have been documented in the literature with regard to considering evidence of practising and pre-service teachers' PCK (e.g., Baker & Chick, 2006; Maher & Muir, 2013). Although this framework identified an extensive set of types of knowledge needed for teaching, and appeared to have potential for use in teacher education (e.g., as done in Chick & Beswick, 2013, 2017), a broader framework was sought for conducting this preliminary examination of teacher educators' knowledge.

Developed from observations of 24 mathematics lessons by Rowland and colleagues, (Rowland et al., 2009), the Knowledge Quartet contains four "units" or dimensions which describe teacher knowledge. Each dimension contains a number of elements that could be used to interpret classroom practice, including that as undertaken by PSTs. An overview of the framework is presented in Table 1.

Category	Description	Elements	Examples of evidence
Foundation	Theoretical background, involving knowledge and understanding of mathematics, knowledge of mathematics pedagogy and beliefs about mathematics	Adheres to textbook Awareness of purpose Concentration on procedures Identifying errors Overt subject knowledge Theoretical underpinning Use of terminology	Concentrate on developing understanding rather than excessively using procedures Show evidence in planning of knowledge of common errors and misconceptions and take steps to avoid them Use mathematical language correctly
Transformation	Ways in which teachers transform or represent what they know for learners	Choice of examples Choice of representation Demonstration	Use equipment correctly to explain processes Select appropriate forms of representations Make use of interactive teaching techniques
Connection	The coherence of the planning or teaching across an episode, lesson or series of lessons; also includes the sequencing of topics of instruction within a lesson	Anticipation of complexity Decisions about sequencing Making connections between procedures Making connections between concepts Recognition of conceptual appropriateness	Make links to previous lessons Make appropriate conceptual connections within the subject matter
Contingency	Teacher's response to unplanned and/or unexpected classroom events	Deviation from agenda Responding to children's ideas Use of opportunities	Respond appropriately to students' comments, questions and answers Deviate from agenda when appropriate

Table 1: Overview of Knowledge Quartet and its elements (adapted from Rowland et al., 2009)

The Knowledge Quartet was designed to be used as a framework for identifying and discussing the ways in which the use of mathematics content knowledge was observed in teaching. In the research reported in this paper, the authors utilised this framework not to discuss the use of mathematics content knowledge in teaching, but rather the PCK for teaching PCK to PSTs. Accordingly, the researchers define content knowledge in this context as including pedagogical as well as subject-matter knowledge, since this is the content knowledge to be taught to PSTs. The research reported in this paper used this framework to examine the teaching practices of two MTEs. Our intention was to reflect upon our teaching

practice, with the assistance of a “knowledgeable other” (Day, 1999) and to interpret this practice through the lens of the Knowledge Quartet. We were motivated to select this framework as Rowland et al. (2009) had developed a range of resources to assist with interpreting the various elements,¹ it has been adopted and reported on by other researchers (e.g., Livy, 2010), and we wanted to determine whether or not it was appropriate for interpreting the work of experienced MTEs, rather than PSTs or in-service mathematics teachers.

Reflection and Peer Observation

It is generally agreed that reflection in, on, and about practice is essential for developing the capacities of teachers (Day, 1999) and reflective practice has been the subject of attention for teacher educators for some time (e.g., Power, Clarke & Hine, 2002). Reflection in this context can be defined as “looking back and making sense of practice, learning from this and using this learning to affect your future action” (Ghaye & Ghaye, 1998, p. 2). Although reflection has connotations of thinking processes and contemplative self-examination, “reflective practice” better encapsulates the detailed analysis that should accompany this reflection (Leitch & Day, 2000). Although many teachers purport to reflect on their teaching practice, Moon (2000) suggests that most do not do so in a deliberate manner which would result in changes in thinking or actions. Sherin and van Es (2003, p. 93) used the term “learning to notice” to describe the process of identifying what is important in a situation, interpreting this in terms of how the important components so identified impact on teaching, and the application of this to more general principles of teaching and learning. Reflective practice, or deliberate reflection, therefore, has a focus on influential factors in events, incidents, and personal experiences. In practice, this involves noticing aspects of one’s own practice that may be triggered by a question from an outside observer, and then recognising and working on issues deemed to be significant. Larrivee (2000) noted that reflection during or simultaneously with actions is often difficult in a busy classroom and therefore requires a perspective from a meta-position, not only to facilitate looking back after the action has taken place but to broaden the perspective via which the action is viewed. This is where an outside observer can assist. According to Brophy (2004), teachers rarely gain new insights or ideas about improving their teaching unless they receive skilful guidance. Self-reflection is limiting as teachers tend to interpret what they observe from their own existing conceptions of effective instruction (Brophy, 2004). To achieve critical reflection, therefore, others are often needed in the process, which is where the help of a mentor or “critical friend” can enhance the reflection process (Day, 1999). The expertise of the critical friend is likely to be a significant factor in determining the outcomes from such observations, and necessary to avoid confirmation of one’s existing beliefs (Schuck, 2002, as cited in Beswick & Chapman, 2012).

Peer observation in higher education settings has been documented in the literature, with results showing that both the observed and the observer have found the experience valuable for and affirming of their own practice (e.g., Hendry & Oliver, 2012). Hendry, Bell, and Thomson (2014) reported on a study of how academics learned about teaching from observing peers. The results showed that the observers particularly benefited from the experience because they were exposed to new strategies or techniques that they were motivated to try out in their own teaching. They also felt reassured about their own practice and less isolated as teachers, because they were able to recognise that they shared similar teaching challenges as their colleagues.

¹ See <http://www.knowledgequartet.org/>

Researching the Work of Mathematics Teacher Educators

Anthony, Beswick, and Ell (2012), document a small number of studies that have contributed to the scant research into the role of the MTE. While it is relatively common to find examples in the literature of teacher educators working with practicing teachers to reflect upon and enhance classroom practices (e.g., Geiger & Goos, 2006; Geiger, Muir & Lamb, 2015; Muir & Beswick, 2007), or examples of teacher educators conducting self-studies of their own practice (e.g., Smith, 2006; Brandenburg, 2008, 2009), the actual role of the teacher educator in mathematics education has been largely unexplored. Maher (2011), in a study involving four teachers seconded as mathematics teacher-educators, identified a number of challenges associated with the role, including the need to link theory with practice, assessment processes, teaching in an online environment, and a feeling that they were not part of the research culture of the institution. While her study provides some insight into the teacher educator's role, it was conducted with inexperienced teacher educators, did not involve classroom observations, and did not have a specific focus on teacher educator PCK.

It is perhaps not surprising that it is generally MTEs who have conducted what little research that has been reported. Much of this research has been self-reflective (Beswick & Chapman, 2012) and limited in terms of being explicit about what was actually learned, or else addressed the issue of how mathematics teacher educators' development could be facilitated by engaging in research on their instructional practices (Beswick & Chapman, 2012; Chick & Beswick, 2013, 2017). Any self-study also needs to recognise the danger that researching one's own practice can become either solipsistic or even narcissistic (Mason, 2008, as cited in Beswick & Chapman, 2012). As Chick (2011) points out, due to the nature of their work, teacher educators end up becoming "god-like arbiters" of what will be included in mathematics education courses as they routinely make decisions about the content, tasks, and emphases that they include in their practice. These decisions are made within the constraints of teacher educator programs, including the number of contact hours available, students' mathematical backgrounds, limited assessment opportunities, and the necessity to address content knowledge before or alongside PCK (Chick, 2011).

Methodology

The aim of the research reported here was to observe teaching episodes in order to note aspects of PCK evident in the work of mathematics teacher educators (MTEs). To achieve this, a case study method was selected due to the appropriateness of case studies when "you [want] to understand a real-life phenomenon in depth, but such understanding encompass[es] important contextual conditions because they [are] highly important to your phenomenon of study" (Yin, 2009, p.18). In this instance, the phenomenon to be examined was the teaching of mathematical pedagogical content knowledge to pre-service teachers, with the context being the teaching of a mathematics pedagogy unit to second-year PSTs in a metropolitan university. For the purposes of this study the case was a unit designed and taught by two mathematics teacher educators that was part of a pre-service primary teacher education program.

In order to examine teacher educator knowledge and actions, the first two authors (Tracey and Jill), elected to work together to observe and analyse their work with students in the second year teacher preparation subject (unit). This unit involved one online lecture each week and, for the students in this study, a two-hour face-to-face tutorial/workshop class to engage in activities to support the lecture material. Tracey was the unit coordinator for the unit, and made most of the decisions about the content to be addressed and activities to be

conducted during the classes. Jill was a tutor for two of the three face-to-face classes (see Table 2), with Tracey taking the third. All three workshop classes were conducted on the same day. The study also involved second-year primary and early childhood PSTs from the three tutorial classes. The unit in this study was the first unit in the degree program designed to specifically address pedagogy for teaching mathematics. The content of the unit included content and pedagogy for the primary mathematics curriculum, with a particular focus on number.

To enhance validity, multiple sources of evidence were identified for collection to enable triangulation of data (Yin, 2009). It was intended that, for each week of the thirteen weeks of the unit, the teacher educators would meet for a pre-observation lesson planning session where a planning pro forma was completed. As part of this meeting the teacher educators identified the goals for the class, the key understandings to be attained, the intended activities and their purpose, and anticipated PST responses to the activities (including likely difficulties).

Each educator then conducted her workshop class (see Table 2). During each class the other acted as an observer. Field notes were made by the observer to record teaching strategies (including those that were responsive to unforeseen dilemmas), activities, teachable moments (key teaching opportunities—planned or unplanned—that could highlight a significant mathematical or pedagogical idea for the PSTs), and any events/ideas that were thought worthy of follow-up discussion. Because of the teaching arrangements for the classes, it made sense to observe the first two classes of the day, since Tracey had only one class and it was convenient and appropriate to observe Jill’s first class. Although Class 3 was not observed, the PSTs were still able to provide feedback as the tutorial was a duplicate of Class 2.

At the completion of each tutorial, the PSTs were given a proforma that enabled them to provide feedback on “something that helped me learn”, “something I found confusing”, and “something that stopped my learning”. The purpose was to identify the educators’ strategies that the PSTs found helpful and to see if the PSTs, who were developing their own knowledge of pedagogy, could identify strategies the teacher had used, either in general or by reference to particular activities, such as “it helped when the tutor used Place Value Charts.”

Within hours of the completion of the classes, the teacher educators met to reflect on the classes. These reflections involved informal discussion that considered the field notes and the student feedback, and resulted in additional notes about the teacher educators’ thinking about the class.

Class	Time (Thursday)	Number of students	Taught by:	Observed by:	Surveys obtained from PSTs
Class 1	9:00 – 11:00	21	Tracey	Jill	yes
Class 2	11:00 – 1:00	25	Jill	Tracey	yes
Class 3	3:00 – 5:00	24	Jill	-----	yes

Table 2: Face-to-face enrolments for ESH319

All data collected—including the planning proforma documents, field notes, observations, reflections, and student feedback—were coded by the first two authors. An adapted open-coding approach (Flick, 2009) was used initially, in which the first two authors drew on their experience and knowledge of existing PCK frameworks (e.g., Chick, et al., 2006; Chick & Beswick, 2013) to identify instances in the data where PCK was used or relevant. The initial coding in this section did not adhere to any one specific framework, and free codes were used so as not to limit opportunities to identify the use of PCK by the MTEs.

Instances of PCK and each author's initial interpretations of the PCK taking place in that instance were compared and discussion was held about the reasons for any differences in perceptions until agreement on the nature of the PCK was reached and a final code applied. The Knowledge Quartet was then used as a framework for organising the final-coded data because it facilitated both broad categorisation and subcategorisation. This process is illustrated with the following example. Several examples of subtraction algorithms that could be addressed informally were intended to be presented in class and these were chosen based on increasing complexity of the mental arithmetic involved: (179-26), (365-47), (800-303). The initial free codes applied to these by the two researchers were "*sequencing – cognitive demand*" and "*sequencing decisions by complexity*". After discussions, these were given the final codes of "*decisions about sequencing*" and "*anticipation of complexity*" and coded under the subcategory "*sequence of topics*" in the Knowledge Building category of "*connection*". Discussions were held with the third author to seek additional feedback as to the interpretation of the categories and descriptors, and the codes applied. This also served the purpose of testing the framework. All data were systematically organised by date then compared for completeness of information. For example, on two occasions, one of the tutors was absent and the observation could not be done. On another occasion, the students had a heavy assignment load and many did not come to class on that particular week.

For the purposes of this paper, the study reports on the classes from one week of the unit. It had the most complete data set to allow the best data triangulation, and incorporated a range of teaching incidents that illuminated teacher knowledge, giving it the potential to provide maximum insight into the two teacher educators' practices. The focus of the chosen workshop was on the structure of different arithmetic problems, such as (a) how removing a subset from a group and, in contrast, looking at the difference in size between two groups both lead to subtraction and (b) that there are different types of division situations known as quotient and partition; and to learn how to model addition and subtraction algorithms with appropriate concrete materials like base-10 multibase arithmetic blocks (MAB).

Results and Discussion

In this section we present the data collected from the pre-observation notes, observation of lessons, pre-service teachers' feedback, and post-lesson reflections for one week's topic. We have elected to focus on one week in order to provide a rich description of the collaborative observation process and the findings. Week 7's topic, "Place value and operating with numbers" was selected as there was a good response rate for PSTs' written feedback, the field notes were particularly rich as a result of the events that occurred in the observed lessons, and the teacher educators' teaching approaches and strategies were similar in both lessons. This provided the opportunity to examine the variation between the classes, since although PSTs' responses can often be predicted for specific activities, the individual/group differences between PST's prior knowledge and experiences resulted in some unpredictability that required the teacher educator to "think on her feet". Responses to these moments offered additional insight into teacher educator PCK.

The data are presented chronologically, beginning with an overview of the pre-observation discussion which also provides a context for the lessons observed. The data related to the lessons observed are then presented, followed by the feedback received from the PSTs. Finally, data related to the post-lesson discussion are presented.

Applying the Knowledge Quartet Framework to Pre-Observation Lesson Planning and Discussion

A few days prior to the weekly lesson, Jill and Tracey met to identify the lesson objectives and to complete the pre-observation pro forma. The lesson objectives that they identified collaboratively were as follows:

- To familiarise PSTs with the different structures of addition, subtraction, multiplication, and division problems;
- To emphasise the importance of exposing school students to a variety of problem structures, and for students to understand the concepts, not just the arithmetic computational processes;
- To explicitly teach the difference between quotient and partition division problem situations;
- To discuss the difference between informal and formal algorithms, and to emphasise the importance of informal algorithms and the implications of teaching formal algorithms;
- To model the addition and subtraction algorithms using appropriate materials such as place-value charts and multibase arithmetic blocks (MAB) (in order to show students how to develop algorithms with understanding), with emphasis on use of correct terminology; and
- To familiarise PSTs with useful tools/materials for teaching place value concepts, such as MAB and number expanders.

Tracey and Jill also identified to each other their intended teaching strategies in order to achieve the learning objectives. These included: the provision of a visual guide for the class to indicate the structure for the lesson (using PowerPoint); the selection of whole number operation problems for PSTs to model based on different situation structures (e.g., the “take away” and “difference” examples for subtraction mentioned earlier); explicit discussion of the difference between quotient and partition division; a directive for PSTs to read an article about algorithm use before class (Clarke, 2005); and, to come ready to engage in discussion about the merits and purposes of informal and formal algorithms; the modelling of formal algorithms using MAB materials and place value charts; and constructing individual physical “number expanders” for highlighting place value and renaming of numbers (e.g, 674 as 67 tens and 4 ones).

The planning proforma had three sections: the lesson objectives, intended teaching strategies, and anticipated PST difficulties/responses. In relation to the third section, Tracey and Jill identified that the following issues might arise for PSTs:

- Confusion with different types of problems, particularly division;
- Pre-conceived ideas and assumptions – e.g., subtract always means “take”; no prior awareness of different types of addition/subtraction situations;
- A tendency to see the equals sign as signalling “do a computation” and/or “this is where the answer goes”;
- Lack of familiarity with the term “algorithm” and the ideas of “informal” and “formal” algorithms;
- A perception that the formal algorithm is the only “right way” of computing an answer, and/or being familiar with only one particular algorithm;
- Inability to see the value of informal algorithms, maintaining a belief that the formal algorithm is the goal;

- Failure (through choice or circumstance) to have read the article prior to class;
- Lack of familiarity with MAB materials and their use;
- The use of inappropriate terms (e.g., “carrying”), perhaps as a legacy of prior learning;
- Difficulty explaining how the algorithm works because they “just do it”; and
- Difficulty with constructing number expanders (e.g., following directions to fold).

As can be seen from the pre-lesson planning, the MTEs’ documentation provided evidence that a number of elements from all categories of the Knowledge Quartet framework were present. *Foundational knowledge*—which, for a teacher educator, involves knowledge of mathematics pedagogy—was evident in their identification of PSTs’ *common errors and misconceptions* about content and pedagogy for primary school mathematics teaching, and in planning for ways to address these. The planned focus on different problem structures showed an understanding of the structural and *theoretical underpinnings* of primary school mathematics content, and the intention to explicitly teach the difference between quotient and partition division demonstrates the correct use of *mathematical language*, as well as recognition that there are different conceptualizations of division. The intention to focus on informal algorithms and modelling with materials also indicated a focus on developing PSTs’ *understanding, rather than relying on procedures*.

In terms of *transforming* the knowledge, the planning documentation shows that the MTEs had determined that MAB materials were an appropriate *choice of representation* for teaching place value concepts. They also intended to make use of interactive teaching techniques as they planned *demonstrations* around the use of the MAB materials along with the provision for the PSTs to practice using the materials and to construct a place value number expander.

The *connection* category of the Knowledge Quartet was evident in the structure, foci, and links across the lesson. Based on their previous experiences with teaching this topic, the teacher educators were confident in *anticipating the complexity* of the topic, and helping the PSTs make *connections among concepts and procedures*. Their lesson notes and the PowerPoint presentation which outlined the structure of the lesson showed a clear *sequence* to the lesson which began with different problem structures before moving on to informal and formal algorithms.

The fourth category in the Knowledge Quartet, *contingency*, refers to the teacher’s response to unplanned and/or unexpected classroom events. It was therefore not appropriate to analyse the pre-planning documentation and discussion for evidence of this; it was expected that this would arise in the lesson observations. Tracey and Jill did, however, discuss some of the issues that they thought may require contingent responses.

Applying the Framework to Lesson Episodes

In the next two sections we illustrate the application of the Knowledge Quartet in the analysis of lesson excerpts from Jill and Tracey’s lessons. The excerpts presented here were selected as they featured commonalities from both teacher educators’ lessons, they related to more than one aspect of the framework, and were particularly illustrative of the aspects that occurred. They also involved the teaching of significant ideas from the content of the unit, and so provide an opportunity to examine the teacher educator knowledge needed to convey that content. Each excerpt was analysed in terms of the four categories of the framework, and these categories frame the discussion and analysis presented here, noting that more than one category was represented in each situation.

Lesson Episode 1: Problem Structures

Using the recommended text as a guide (Van de Walle, Karp & Bay-Williams, 2013), PSTs were directed to work in pairs to use counters to represent different problem structures, such as modeling “join” problems where the initial quantity, change, and result were, in turn, unknown. In this part of the tutorial, the MTEs worked to ensure PSTs remained on task, questioned PSTs about their understanding of the task, and provided guidance for those who may not have completed the task correctly. The task was relatively straightforward and did not reveal any misunderstandings from the PSTs. Following PSTs’ whole group sharing of the task, the MTEs emphasised that the task illustrated the knowledge required by a teacher, namely to be able to recognize and construct different problem structures, rather than being directly intended as an activity for children to complete. Common student errors and misconceptions associated with the operations were also discussed, such as “subtraction does not always involve ‘taking away’” and that “the equals sign is not a signifier for where the answer goes”.

Quotition and partition division situations were then discussed with the PSTs and they were then asked to individually write story problems which demonstrated quotition and partition division scenarios. This proved more challenging, with some PSTs unable to compose a quotition example without assistance, and one PST providing a multiplication example: “Tom had 12 bags of lollies with 4 lollies in each bag. How many lollies did he have?” As some PSTs still seemed confused about distinguishing the difference between the two situations, more PSTs were selected to share their examples aloud to the class. Tracey chose to respond to this by giving a practical example. In order to demonstrate why quotition division can be a more appropriate “action” than partitioning in some circumstances, Tracey chose to share the example of $1\frac{1}{2} \div \frac{1}{4}$ on the board to the class, highlighting that it does not make clear sense to talk about partition ($1\frac{1}{2}$ objects shared among $\frac{1}{4}$ of a person), but that a quotition interpretation is more meaningful (asking how many orange quarters can be made from $1\frac{1}{2}$ oranges).

Foundation

Tracey’s decision, as unit coordinator, to expose PSTs to the different problem structures was informed by her professional knowledge of the theoretical underpinnings for arithmetic problem structures. In particular, the structures—and the resulting arithmetic solutions—are based on the kinds of relationships involved amongst the quantities (Carpenter, Fennema, Franke, Levi, & Empson, 1999). In her explanation to the PSTs while in the classroom, she directed them to the appropriate pages in the textbook and emphasized the correct use of mathematical terminology. When monitoring the PSTs’ attention to the task she noticed that many seemed to be carrying out the procedures mechanistically and she was concerned that some may not have realized the importance of exposing children to different problem types, so that children would encounter all the different types of arithmetic problems (e.g., addition calculations can arise from what looks like an subtraction situation: “I have some cards and, after losing 17, I now have 22; how many did I have at the beginning?”). One PST also asked about doing the problem structures activity with school students, providing Tracey with the opportunity to make the class aware of the purpose of the activity: it was to help them as teachers learn about different problem types, to ensure that they would give children a range of situations when constructing arithmetic word problems. Her individual monitoring of PSTs’ ability to compose different division problems also demonstrated Tracey’s subject knowledge, in that she was able to recognize appropriate examples and assist those PSTs who were having difficulty.

Transformation

Tracey's own knowledge of the different problem structures was secure, but the challenge she faced was transforming this knowledge into a form that could be accessed by and be relevant for PSTs. Her choice of examples and representations was guided by the textbook, hence demonstrating a belief in the authority of this work (an aspect of foundation knowledge). The following example was given for a "Join: Change Unknown" situation: "Sandra had 8 pennies. George gave her some more. Now Sandra has 12 pennies. How many did George give her?" All of the addition and subtraction examples had an American context yet Tracey did not acknowledge this. While this may not have impacted upon the PSTs' ability to carry out the task, it may have impacted upon their overall engagement and perception of relevancy.

After providing a demonstration of how to use counters to respond to the problem structures task, she provided the PSTs with the opportunity to work through the examples at their own pace, indicating a belief in using immersive teaching strategies. Interestingly, this aspect is not explicitly mentioned in the Knowledge Quartet, although it is present in Chick et al.'s (2006) framework. Counters were recommended for use in the text and Tracey recommended using different colours to represent the respective parts, hence demonstrating an appropriate choice of representation. PSTs' evaluation of the lesson showed that they identified the modelling with materials and the opportunity to "physically do the activities" as being "helpful to their learning".

Tracey relied upon the PSTs to provide examples of partition and quotient situations, although the pre-planning pro forma shows that the teacher educators were aware these may cause confusion and had included examples in the on-line lecture materials that were available to the PSTs before the workshop. Post-lesson reflections between Tracey and Jill showed that both were concerned that some of the PSTs may not have developed a sound understanding of the difference between partition and quotient division. Time management was a factor in addressing this, as both MTEs felt required to cover the intended content and to move on in the lesson before being convinced that the concepts were understood.

Tracey did, however, deliberately choose $1\frac{1}{2} \div \frac{1}{4}$ to demonstrate the applicability of quotient division when operating with certain fractions. The numbers in the problem were selected because they were familiar fractions and allowed for the focus to be on the operation or action, rather than using complex fractions. Through a diagram representing a length of ribbon one and a half metres long, Tracey demonstrated that lengths of $\frac{1}{4}$ of a metre could be made through repeated subtraction. Jill, who observed Tracey's lesson prior to taking her own lesson, thought this to be a particularly effective example and used it with her own PST class.

Connection

Mathematics educators face the dual challenge of making connections within and among mathematical concepts, and making connections between mathematics education theory and classroom practice. In this particular lesson excerpt, there were opportunities to connect aspects of mathematics, such as the links between addition and subtraction, multiplication and division, and division as repeated subtraction. The Knowledge Quartet explicitly mentions "making connections between procedures" and "making connections between concepts", and we would recommend including "making connections between theory and practice" when interpreting the work of teacher educators. There were a number of incidents in the lessons observed when both Tracey and Jill used a think-aloud technique

to expand on the reasons for their teaching approaches. For example, Tracey explicitly explained that the modelling of problem structures with counters was aimed at increasing the PSTs' own knowledge, rather than an activity that would be enacted exactly as they performed it in a school classroom.

In terms of making connections among concepts, and anticipating its complexity, this topic (in particular, quotient and partition situations) may have been under-estimated in its difficulty by the teacher educators, as some PSTs were still confusing the different division structures by the end of the activity. As previously indicated, however, time constraints meant that Tracey felt compelled to move on to the next part of the lesson (informal and formal algorithms) before ensuring all PSTs had made the necessary connections.

Contingency

As can be seen from the pre-planning documentation, Tracey and Jill had made some reasonable predictions about PSTs' previous experiences and likely difficulties with the subject matter. It was not, however, anticipated that there would be confusion between multiplication and division. However, a teachable moment occurred when a PST wrote a multiplication story problem, rather than a division one. Tracey addressed this through referring to the language used of "how many" and cautioned against the tendency to make the phrase "how many" synonymous with division. She also attempted to elicit reasons for this from the PSTs, which prompted another contingency example. When the PSTs were asked as a whole group about why it might not be a good idea to equate "how many" with "division", no one responded, leaving Tracey to think, "Do I phrase the question in another way?", "Do I ask a class member and risk putting them on the spot?", or "Do I give them my opinion/answer?" In this instance she essentially kept rephrasing the question until someone responded and followed this up with her own rationale which involved discussing the (limiting) teaching practice of looking for key phrases in word problems.

Lesson Episode 2: Informal and formal Algorithms

Prior to the commencement of the tutorial, students were requested to read Clarke's (2005) paper on the teaching of written algorithms in the primary years. In this paper, Clarke detailed the dangers of introducing formal algorithms to students too early in primary schooling, particularly before solid understanding of related concepts and strategies are developed. The purpose of having students engage in the pre-reading was so that they could become familiar with Clarke's arguments. It was expected that these arguments would, for the most part, contrast with and challenge the PSTs' own rule-based experiences with school mathematics.

In the workshop class, decisions about sequencing were made carefully to establish this concept, beginning with providing the PSTs with number problems to calculate informally. After presenting the first problem (179-26) and asking PST's to solve it using an informal algorithm, it became quickly apparent from the resulting responses and confusion that the PSTs did not understand the difference between a formal and an informal algorithm. Both Tracey and Jill experienced similar responses and both asked the PSTs whether they had read the assigned article. It transpired that no student in either group had, limiting their understanding of the work to be undertaken and the discussion to be had around the introduction of formal algorithms. In the case of Jill's class, Jill provided further clarification and introduced some additional problems that allowed students to start thinking about

different ways to calculate numerical problems informally. Tracey also provided additional examples to distinguish between the nature of informal and formal algorithms.

Foundation

Jill and Tracey had some familiarity with their PSTs' prior experiences and beliefs about mathematics as this was elicited both explicitly and implicitly earlier in the unit. A majority of the PSTs had experienced school mathematics that had focused heavily on the use of formal algorithms. In planning for this week's workshop, Jill drew on her knowledge of learners in recognizing that shifting students away from a reliance on formal algorithms would likely be difficult and therefore learning would need to challenge existing ideas quite strongly. Her recognition of this as a concern gives insight into Jill's own beliefs about mathematics and what constitutes valid mathematical activity, experiences, and processes. The focus on the necessity for children to develop understanding in mathematics prior to learning formal procedures, suggests a view of mathematics as conceptual rather than procedural, thus demonstrating a focus on *developing conceptual, rather than procedural understanding*.

Transformation

The challenge of teaching is often for the teacher to take what is known and to transform this into knowledge that can be accessed by students (Rowland, et al., 2009), in this instance, PSTs. The intent was to familiarize the PSTs with informal and formal methods of calculation, and to then contrast the methods to demonstrate how informal methods draw on mathematical fluency and familiarity with the meanings of operations and the associated mathematical relationships. Formal methods are often accurate but, once taught, can be overused and often mask the underlying mathematical structures that are drawn upon. Rather than simply telling PSTs the advantages of extending work with informal algorithms, Jill sought to demonstrate to PSTs, through their own involvement, the way in which informal approaches draw on mathematical structures more so than procedural approaches. Although Jill was aware of this knowledge herself, decisions had to be made about how to *transform* this knowledge in such a way that PST's could experience it for themselves. Jill chose to do this by putting up examples of relatively simple number problems on the board (for example, 14×4 and $146 - 50$). These problems were carefully chosen so as to have multiple possible methods for informal calculation. The PSTs were asked to work these out informally and then, after being given a few moments, were asked to share methods until such time as all ideas had been exhausted. For example, 14×4 could be calculated by multiplying 15×4 and then subtracting 1×4 (or 4). It could also be calculated by multiplying 10×4 and adding 4×4 ; or by doubling 14 and then doubling that answer again; or by doubling 4 and halving 14 to get 8×7 . Each of these possible responses makes use of key underlying mathematical structures, such as place value and the distributive law, and demonstrates their importance to arithmetical computation. These methods were able to be discussed along with the relevant structures. In this way, PSTs not only were able to unpack the mathematical structures involved, but also have the process *demonstrated* to them for use themselves in their role as teachers. This was then contrasted with the formal algorithm which, while deriving a correct answer, relies on procedural methods and is quite restricted in its opportunities for such discussions. With this approach, Jill sought to transform a highly contestable topic into

something PSTs could see and experience first-hand, strengthening the case for the use of informal approaches.

Connection

In order to maximize the connections among the role of algorithms, the mathematical structures that underpin them (e.g., the distributive law), and the when and how of teaching them, the order of content in the lesson was carefully planned. Initially, students were introduced to addition and subtraction problem structures: change problems, compare problems, and part-part-whole problems (Van de Walle et al., 2013). This was to allow a segue into a discussion on introducing children to informal and formal algorithms and addressing issues associated with introducing formal algorithms to children potentially before their conceptual understanding of operations and structure was solid. Finally, it was expected that this discussion would draw out the need for careful conceptual development, and the use of appropriate strategies, representations and language, which would set the scene for hands on work with MAB blocks, place value charts, and number expanders.

Contingency

Much that occurred in this workshop hinged on the PSTs having engaged with the reading material so that the terminology of “formal” and “informal” algorithms, and the arguments for using informal approaches had already been encountered. When it was determined by both Jill and Tracey that the PSTs had not engaged with the reading, this created a quandary. During the planning stage, the MTEs had considered that some students may not have done the reading, but not that no-one would. This created a situation where there was not even a critical mass to carry the conversation. Both Tracey and Jill were put in a situation of needing to make a rapid, unanticipated response in terms of addressing both the examples students were being asked to provide, and the intended discussion about introducing formal algorithms.

Jill and Tracey both responded similarly in the first instance. They took one of the intended examples and carefully modelled both formal and informal approaches to solving, and then continued by having the PSTs address the other intended examples and sharing and discussing responses as intended. However, the more difficult issue stemmed from the inability to hold the planned discussion around the introduction of formal algorithms. Tracey approached the difficulty as a “teachable moment” (Clarke, Cheeseman, Gervasoni, Gronn, & McDonough, (2002) and asked the PSTs how they would deal with this in their own classes if their students were unprepared for the requirements of their own learning. She then proceeded to discuss the content of the paper but, due to time constraints, was unable to engage in the final activity which was to create a number expander and demonstrate its use. Jill also used the difficulty as a “teachable moment” but responded in a different manner by explaining the consequences of the situation and articulating the decision she now had to make as an educator. She explained to the PSTs that by not having read the article, she was now in a difficult situation as a teacher. She had to decide how to proceed. She explained what she believed the options to be: to unpack the article and explain the content and hold the intended discussion, and thus miss an important component of the class—the making and using of number expanders—or she could avoid the article, expect them to read it in their own time and hope they would, and continue with the number expander activities. She explained that she would choose the latter because the students could read the article in future but would not have the opportunity to again address number expanders. She stressed that she

felt the contents of the article were more important, because they impacted critically on pedagogy but that she was making the decision based on utility and the fact that the PSTs could not easily work with number expanders in their own time.

PST Feedback

In this section the PSTs' own identification of teacher educator knowledge is examined, using the feedback that some of the students provided to the workshops. A total of 16 PSTs provided feedback on Jill and Tracey's lessons. In response to "Something that helped me learn," 11 PSTs made reference to the use of visual representations and modelling with materials, showing alignment with the *transformation* category of the Knowledge Quartet. Closely related to this was the reference to physically carrying out the activities, demonstrating Jill and Tracey's belief in making use of interactive teaching techniques. Illustrative comments included the following:

Using the blocks to teach addition and subtraction, especially with larger numbers [PST701]

Physically doing the work ourselves [PST703]

The number expanders and the PVC [place value chart] sheets were really good and helped me think about how students would solve the problems [PST713]

There were three references made in relation to discussions and/or explanations, which linked with the *connections* aspect of the Knowledge Quartet.

With regard to identifying "Something that stopped my learning", seven PSTs identified "nothing", while the others mentioned becoming confused with using the MAB materials to model subtraction when regrouping was required, their own background in terms of understanding the mathematics required, and the use of incorrect terminology. Five PSTs also indicated that they were still unsure about the difference between quotient and partition division and three expressed confusion between informal and formal algorithms. Five PSTs explicitly mentioned not reading the Clarke article before class, likely influenced by Jill's strong emphasis on this, and one identified "learning other ways to calculate". In terms of connections with the Knowledge Quartet, reference to the use of materials and explanations aligned with the *transformation* category.

Post-lesson Reflection

Within hours of observing each lesson, Jill and Tracey met to reflect on what they had each observed and to clarify aspects of the practices observed. Much of the discussion focused around the issue of the PSTs not having completed the pre-reading of the Clarke article and how this impacted upon the intended lesson objectives and learning outcomes. As previously explained, the way in which they responded to this differed. In the post-lesson discussion, it was generally agreed that Jill's response to use that as a "teachable moment" and spend time discussing what to do when students come to class unprepared, was probably the more appropriate approach. This enabled Jill to proceed with covering the other lesson objectives and intended teaching strategies. In this case, though, it must be noted that what the MTEs were teaching was a general pedagogical principle, rather than a mathematical or maths pedagogy principle, which reveals the diversity of issues that must be addressed in teacher education units.

Both Jill and Tracey noted that, in terms of content, both groups had difficulty with distinguishing between quotient and partition division and between informal and formal

algorithms. There was also a lot of scaffolding required when the PSTs were modelling the algorithms using the MAB materials. The frequent use of inappropriate terms, such as “carrying” and “borrowing,” was noted, along with a lack of understanding how the written recording reflected what was happening with the regrouping of the MAB materials.

In terms of reflecting upon the lessons with regard to their own practice, several common themes emerged. There were a number of incidents in each lesson when contingency decisions were required. These mostly occurred when PSTs were asked questions in a whole class situation and no-one responded. They also occurred when PSTs’ content knowledge impacted upon their ability to engage with the activities. The difference between quotient and partition division problems was an example of this. Both educators observed incidences of “talking aloud” to the PSTs about their own teaching decisions and/or making teaching approaches explicit. Both Jill and Tracey felt that having each other in the room while the lesson took place also assisted with this, as a dialogue could also occur between them.

Conclusions and Implications

The work of the teacher educator is complex, especially when the content of learning involves both subject discipline knowledge (e.g., mathematics, as in the case here) and appropriate pedagogical content knowledge. Some of the educator’s work, particularly dealing with general pedagogies and classroom management, is analogous to that of a school teacher, as seen in the second episode where Jill and Tracey had to deal with students who had come to class unprepared. Just as in regular teaching, the decision about how to proceed was not merely a matter of addressing the behaviour, but of making strategic choices about how to best achieve the intended learning goals for the lesson in the light of students’ readiness to proceed. Other aspects of teacher educator work seem to be less straightforward, precisely because the learning objectives can be a non-separable mix of pedagogical content knowledge and discipline knowledge. An example of this occurred when Tracey sought to help the PSTs understand about different problem structures as well as helping them to understand why teachers need to know this but not necessarily explicitly teach the different types in their classrooms. This also provides an example of the additional skills and knowledge required by a MTE. As suggested by this discussion, teacher educators must know PCK for teaching mathematics for two reasons: because they need to teach aspects of mathematics to the PSTs just like a school mathematics teacher, but also because PCK for teaching mathematics must itself be taught to the PSTs. In answer to the first research question, the results and discussion demonstrate that several aspects of MTEs PCK was evident including choice of examples, choice of representation, demonstration, anticipation of complexity and responding to PSTs’ ideas. The talk aloud technique employed by both MTEs was the dominant strategy for making these PCK aspects explicit for the PSTs.

The Knowledge Quartet appeared to be useful as a broad framework for categorising the types of knowledge that Tracey and Jill brought to bear in their teaching. The finer grained descriptors from the third and fourth columns of Table 1—which echo some of the categories in other knowledge frameworks—were useful in allowing the identification of particular examples of knowledge being put into use. The categories provided a structure for both the observer and the educator to attend to knowledge use in teaching, and seemed to capture and describe what appeared to be the critical moments in planning for and carrying out teaching. It appears, then, that the Knowledge Quartet—originally designed for the teaching of mathematics—generalises “upwards” to the mathematics educator’s work of

teaching PCK for the teaching of mathematics, with the proviso that the “content” to be taught is now more multifaceted (as already discussed in the previous paragraph).

In just these two workshops, each of the three enacted knowledge components was vital to Tracey and Jill’s work. Transformation was essential as Tracey tried to make arithmetic problem structures understandable both as a concept in its own right and as an underpinning framework for constructing arithmetic tasks for students. Connection was evident as both educators planned the lesson to cover types of operations and then the nature of algorithms for computing the results of arithmetical operations. Contingency—which usually requires the teacher to construct a solution to an unexpected teaching dilemma on the spur of the moment and is thus the component of the Knowledge Quartet that is hardest to predict and measure—was demonstrated by Jill as she took advantage of the PSTs’ failure to do the reading by working through some additional examples to highlight computation strategies but also as she highlighted more general teaching issues by discussing the reasons for her choices as a teacher when confronted with the very dilemma raised by the PSTs’ lack of preparation.

As was the case when Chick and Beswick (2013, 2017) analysed the work of the third author interacting with PSTs in an online environment, contingency seems particularly critical to the work of the educator. The capacity to make decisions “in the moment”, constructing possible solutions for an unexpected teaching situation and then choosing and justifying one solution as optimal, is vital. As Mason and Davis (2013; see also Mason, 1998) suggest, educators need *awareness* of the content and context of teaching. They propose that this involves not necessarily knowing *more* but *noticing* more (so that the situation is better understood), and knowing more *deeply* (so that dependencies and interrelationships are identified). This allows a greater range of possible actions—mathematical and pedagogical—to come to mind in a contingent situation, to be weighed up, chosen among, and acted upon. They write of the importance of being “with” the content knowledge of teaching to know it deeply, and to see that content knowledge in ways that are helpful for learners. For teacher educators this means having deep understanding of PCK—the content domain for teacher education units—and being able to see and present that knowledge in ways that are helpful for learners, and it frequently means developing it “in the moment.”

In the complex space of teacher education an examination of the PCK held by teacher educators allows debate about what is critical to teach and how best to teach it. Frameworks such as the Knowledge Quartet make it possible to unpack the complexity of the work of the teacher educator. This study has provided only a case study snapshot of the knowledge and actions used in the work of preparing PSTs, and further work is needed to be more confident of its representativeness. However, the authors’ experiences in other pre-service teacher education work and the provision of professional learning programs suggest that PCK frameworks provide a valuable way for characterising our work. It would, nevertheless, be worth investigating more deeply the ways in which contingency depends on held knowledge and what processes take place as in-the-moment teaching decisions are made.

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