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Abstract : This paper considers a ew class of time series models called Autoregressive Conditional Duration (ACD)models.Various statistical properties of this class of ACD models are given. A minimum mean square error (mmse)forecast function is obtained as it plays a important role in many practical applications.The theory is illustrated using a potential application based on financial data.

Keywords: *Autoregressive, Conditional expectation, Intensity, Hazard function, Stochastic process, Prediction, Estimation, Irregular data,Transaction data, Finance, Autocorrelations.*

1. Introduction

In traditional time series analysis investigators are concerned with the behaviour of the variable of interest (i.e. price, volume, temperature etc) separated by equally (or unequally)spaced fixed time points.That is, in this case, the time process is considered as being non-stochastic.The general time series theory of Auto-regressive Moving Average (ARMA) (see Box and Jenkins (1976)) or some of its modifications (see Brockwell and Davis (1991)) can be used in the modelling and forecasting of such situations. Although many financial data may be treated as time series, the standard techniques of time series analysis cannot be employed here directly due to the rapid variation of the time intervals. Since many finance problems involve the arrival of events such as prices or trades in irregular time intervals, a new direction of modelling is necessary to explain the properties of such data.With that view in mind Engle and Russell (1998) introduced a new class of models called "Autogressive Conditional Duration "(ACD) models. The formulation of this ACD class of models focuses on the inter-temporal correlations of the durations or the time intervals between events. In Section 2, we review this ACD class of models in order to form a basis of this paper.

2. ACD Models and their Properties

Transaction data can be described by two types of random variables. One is the time of the transaction and the other is the observation (called marks) at the time of the transaction. It is well know that financial data inherently arrive in irregular time intervals and investigators are concerned not only with the variable of interest (ie. price, quote, volume), but also the time of each incident. Thus in many financial modelling problems, the variable 'time 'is considered as stochastic (or random) and the corresponding analysis proposes an alternative method to the traditional fixed time (or interval) analysis. Now the statistical problem is to estimate the probability of a event, for example, the price at each time point. This requires specifying the stochastic process of arrival times, estimating the parameters and computing the probability of events.The instantaneous probability of an event is called the intensity of the process. In dependent processes this intensity is obtained by conditioning on past information.

Consider a sequence of arrival times T_1 , T_2 , $\cdots$,T_N from a particular point process, where T_i is the time at which the i^{th} trade occurs and $T_1 < T_2 < T < \cdots < T_N$. Suppose that the observation at time T_i is denoted by X_i. Denote by F_i the σ field generated by all random variables $\{(T_i , X_i); 1 \le i \le N \}$. Let N (T) be the number of transactions (or events) occurring by time T . Obviously, N (T) is a (non-decreasing) step function of time with N $(T_0) = 0$. (N(T) is continuous from the left with limits from the right). Define the conditional intensity of a process λ as

$$\lambda(T|N(T),T_1,\ldots,T_{N(T)})=\lim_{\Delta T\to 0}\frac{P(N(T+\Delta T)>N(T)|N(T),T_1,\ldots,T_{N(T)})}{\Delta T} \qquad (2.1)$$

A function similar to that of (2.1) is used in survival analysis and is called the hazard function (see, for example Kalbfleisch and Prentice (1980)).

Let D_i be the interval between two events (or waiting times) for the events at T_i and T_{i-1} such that

$$D_i = T_i - T_{i-1}\ ;\ i = 1, 2, \cdots, N. \qquad (2.2)$$

Note that the values of D_i are the i^{th} duration between i^{th} and $(i-1)^{th}$ trades.
Consider the conditional expectation of D_i as give below:

$$E(D_i \mid D_{i-1}, \ldots, D_1) = \psi_i \qquad (2.3)$$

where ψ_i is a function of $D_{i-1}, \cdots, D_1$ and Θ is a vector of parameters such that

$$\psi_i = \psi_i(D_{i-1}, \ldots, D_1; \Theta) \qquad (2.4)$$

It is obvious that $\psi_i > 0$ since $D_i > 0$.

A new class of models for possibly unequally spaced correlated data (example, financial data) is developed via the dependence of the conditional intensity on the past durations. The crucial assumption for this class of models is that the dependence can be summarized by a function ψ_i (the conditional expected duration give past information) with the property that ψ_i / D_i are independent and identically distributed (iid) random variables. Equivalently,write

$$D_i = \psi_i e_i, \qquad (2.5)$$

where e_i's are iid random variables with the probability density function $P_0(e;\Phi)$, which must be specified, and Φ and Θ are variation free. Further assume that e_i's are independent of D_i and $E(e_i | F_{i-1}) = 1$. Since the durations and expected durations are positive, the multiplicative disturbance naturally will have positive probability only for positive values and it must have a mean of unity.This assumption requires that all the temporal dependence in the durations be captured by the mean function. This assumption is testable in practice using the standardized durations.

A new class of models is developed based o the parameterizations of (2.3) and (2.5). It is clear that the probabilistic structure of D_i is similar to that of an autoregressive (AR) process and hence the class of models described by (2.3) and (2.5) are called autoregressive conditional duration (ACD) models since the conditional expectations of the durations, D_i, will depend upon the past durations D_{i-1}, D_{i-2}, …….. D_1 as in the AR situation.

It is possible to define a family of ACD models satisfying (2.3) and (2.5) via different specifications for ψ and for the distributions of e.

Define

$$S_0(t) = P_0(e \geq t),\ t > 0, \qquad (2.6)$$

where $S_0(t)$ can be considered as the survival function associated with $\{e_t\}$.

Clearly, $\lambda_0(t) = \dfrac{P_0(t)}{S_0(t)}$ can be considered as the baseline hazard function since it does not depend upon any conditioning information.
Now we state an important result from Engle and Russell (1998).

Result 1: The conditional intensity for an ACD model based on (2.3) and (2.5) is given by

$$\lambda(T|N(T),T_1,\ldots,T_{NT}) = \lambda_0\left(\frac{T - T_{N(T)}}{\psi_{N(T)+1}}\right)\frac{1}{\psi_{N(T)+1}} \qquad (2.7)$$

This indicates that the past history influences the conditional intensity by both a multiplicative effect and a shift in the baseline hazard function.
It is easy to verify that when the durations are conditionally exponential, the baseline hazard function is unity and in this case the conditional intensity in (2.7) reduces to

$$\lambda(T|N(T),T_1,\ldots,T_{NT}) = \frac{1}{\psi_{N(T)+1}}$$

Now we consider some general forms of the ACD class in Section 3.

3. General ACD Models

Consider the class of models given by

$$\psi_i = \omega + \sum_{j=1}^{p}\alpha_j D_{i-j} + \sum_{j=1}^{q}\beta_j \psi_{i-j}\ ,\ i = 1,2,\ldots,N \qquad (3.1)$$

where $\omega > 0$ and α_j and β_j are non-negative constants, $D_0 = \psi_0 = 0$ and p and q are the orders of the corresponding lags.

This is called a ACD (p,q) memory model or simply an ACD (p,q). This model (3.1) implies that only the most recent p actual durations and recent q expected durations influence the conditional mean durations. This model (3.1) introduces an ACD family with infinite memory specifications of the intensity.
Let $\xi_i = D_i - \psi_i$. Now (3.1) reduces to

$$D_i - \xi_i = \omega + \sum_{j=1}^{p} \alpha_j D_{i-j} + \sum_{j=1}^{q} \beta_j (D_{i-j} - \xi_{i-j})$$

or

$$D_i = \omega + \sum_{j=1}^{p'} (\alpha_j + \beta_j) D_{i-j} - \sum_{j=1}^{q} \beta_j \xi_{i-j} + \xi_i \quad (3.2)$$

where p' =max (p, q).

Clearly, (3.2) is a ARMA (p, q) type model with highly auto-correlated innovations.

NOTE: When q = 0, (3.1) reduces to

$$\psi_i = \omega + \sum_{j=1}^{p} \alpha_j D_{i-j} \quad (3.3)$$

This is a simple p-memory specification of the intensity. In this case the most recent p durations influenced the conditional duration ψi. Obviously, (3.3) reduces to an AR type model with autocorrelated innovations satisfying

$$D_i = \omega + \sum_{j=1}^{p} \alpha_j D_{i-j} + \xi_i \quad (3.4)$$

Equations (3.2) and (3.4) are useful ARMA type models for durations. Forecasts for waiting times can be obtained from these representations using the standard ARMA theory. For example, the one-step-ahead mmse forecast function, $D_T(1)$ of D_{T+1} based on (3.2) (assuming α_j and β_j are know) is

$$D_T(1) = \omega + \sum_{j=1}^{p'} (\alpha_j + \beta_j) D_{T+1-j} - \sum_{j=1}^{q} \beta_j \xi_{T+1-j} \quad (3.5)$$

The one-step-ahead mmse forecast function, $D_T(1)$ of D_{T+1} based on (3.4) is simply

$$D_T(1) = \omega \sum_{j=1}^{p'} (\alpha_j + \beta_j) D_{T+1-j}. \quad (3.6)$$

However, the model (3.1) is convenient in theoretical development as it allows various moments to be calculated easily. From (3.1), it is obvious that the un-conditional mean of D_i is

$$\mu = E(D_i) = \frac{\omega}{1 - (\sum_{j=1}^{p} \alpha_j + \sum_{j=1}^{q} \beta_j} \quad (3.7)$$

where $0 < \sum \alpha_j + \sum \beta_j < 1$.

For example, when p = 1 = q, the corresponding standard ACD (1,1) specification is given by

$$\psi_i = \omega + \alpha D_{i-1} + \beta \psi_{i-1}. \quad (3.8)$$

The unconditional mean in (3.7), in this case reduces to

$$\mu = \frac{\omega}{1 - (\alpha + \beta)} \quad (3.9)$$

The corresponding unconditional variance of this ACD (1,1) in (3.8) is

$$\sigma^2 = \frac{\mu^2 (1 - \beta^2 - 2\alpha\beta)}{1 - \beta^2 - 2\alpha\beta - 2\alpha^2}, \quad (3.10)$$

provided $\alpha+\beta<1$, $\beta^2 + 2\alpha\beta < 1$, and $\beta^2 + 2\alpha\beta + 2\alpha^2 < 1$.

In this case it is easy to verify that

$\frac{1 - \beta^2 - 2\alpha\beta}{1 - \beta^2 - 2\alpha\beta - 2\alpha^2} > 1$ and hence $\alpha > \mu$. This exhibits the excess dispersion of the variable D_i as often noticed (in econometrics) in duration data.

Although this ACD (1,1) given in (3.8) seems to be a very useful member of this class of models in practice, in Section 4,in brief, we describe the identification and estimation procedures in general.

4. Parameter Estimation

Specifications of (3.1) can be generalized in many ways using different distributions for e_i. In view of the non-negativity of D_i, a natural (and more popular) form of the distribution of e_i is the Weibull with the probability density function

$$f(\theta) = \gamma \kappa^{\lambda} \theta^{\lambda-1} \exp\{-(\kappa\theta^{\lambda}\} \quad (4.1)$$

where $\theta \geq 0$,and $\kappa,\gamma > 0$.

The failure rate function or the baseline hazard function associated with (4.1) is

$$\lambda_0(\theta) = \gamma \kappa^{\gamma} \theta^{\gamma-1} \quad (4.2)$$

From (2.7), the conditional intensity in terms of Ψ_i is

$$\lambda(T|N(T), T_1, \dots, T_{NT}) = \gamma\{\Gamma(1 + 1/\gamma)\psi^{-1}_{N(T+1)}\}^{\gamma}\{T - T_{N(T)}\}^{\gamma-1} (4.3)$$

where $\Gamma(.)$ is the gamma function.

When $\gamma = 1$, (4.3) reduces to the conditional intensity for the exponential case.

The parameters of (3.1) can be estimated by maximizing the corresponding log likelihood function

$$\sum_{j=1}^{N} \log(\lambda/D_i) + \gamma \log(\Gamma(1 + 1/\gamma) D_i/\psi_i) - (\Gamma(1 + 1/\gamma) D_i/\psi_i)^{\gamma} \quad (4.4)$$

[See Allen, McDonald and Yang (2001)].

To evaluate (4.4), one needs some parameterization of ψ_i . For example, for an

ACD (1,1) (as in (3.8)), ψ_i, $i > 1$, is recursively obtained with the initial value $\psi_0 = \alpha$.

5. An Application

The data sets used in this paper are based on a sample of high frequency transactions data acquired from the Securities Industry Research Centre of the Asia-Pacific (SIRCA). The quote prices are viewed trade by trade for a listed Australian company, News Corporation from the Sydney Stock Exchange over a period of sixty-two trading days in the first quarter of 2000. As suggested by Lee and Ready (1991), for every transaction the prevailing quote is the last quote which appears at least five seconds before the transaction itself. All transactions that occurred from 10:10 am to 4:00 pm are adopted for every trading day. The first ten minutes trading at the opening and the overnight price change are removed to avoid the influences of overnight news arrival. News Corporation has an average market capitalisation of 37.8 billion dollars and an average share price over the sample period of A$21.96. Summary statistics for this stock and the ACD(1,1) estimates using (4.4) are presented in the Table below:

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