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Investigating the Different Dimensions of Preservice Mathematics Teachers' Understanding – The Case of Factorisation

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Abstract: This paper reports on a study designed to investigate preservice teachers' understanding of factorisation, a topic not explicitly taught within their teacher education programme, but one they will be required to teach when they graduate. We query if the knowledge they bring from secondary school, prepares them sufficiently to teach their future students for understanding. 83 preservice secondary school mathematics teachers' procedural and conceptual understanding of quadratic factorisation were assessed using Usiskin's Framework for understanding mathematics (2012) which identifies several dimensions of understanding. The study provides evidence that the preservice mathematics teachers have a strong procedural understanding, and while some conceptual understanding does exist, there was very limited conceptual understanding within most of the dimensions of the framework (Usiskin, 2012). We conclude the paper by considering how teacher educators can address the issues of preservice teacher knowledge and understanding of content not formally covered within their teacher education programmes.

Background to Research

There is widespread consensus on the need for a teaching for understanding approach to mathematics (Blumenfeld, Marx, Patrick, Krajcik, & Soloway, 1997; Eisenhart et al., 1993) and this has resulted in a growing demand on the mathematical knowledge and understanding of teachers (Selling, Garcia, & Ball, 2016). Students frequently depend on algebraic procedures (Dufour-Janvier, Bednarz, & Belanger, 1987), yet conceptual understanding can be achieved only through multiple representations and making connections to other concepts (Kotsopoulos, 2007; O'Meara, 2011; Hourigan and Leavy, 2019). Mathematics teachers therefore require sufficient understanding in order to foster this level of understanding in their students, but the findings of Ball (1990b), Hannigan et al. (2013), and Fitzmaurice et al. (2019) indicate that the conceptual understanding of preservice teachers is inadequate for this approach to teaching mathematics. Borko et al. (1992) and Slattery and Fitzmaurice (2014) suggest that this has implications for their teacher education programs. A significant amount of the literature examining teacher understanding focuses on primary/elementary teachers' understanding (Borko et al., 1992; Crespo & Nicol, 2006; Holm & Kajander, 2012; Tirosh, 2000; Hourigan and Leavy, 2017; Hourigan and Leavy, 2019) rather than preservice secondary school mathematics teachers which are the focus of this research. There further exists a considerable cleft in the literature on the concept of factorising and understanding of factorisation, a topic that almost all preservice teachers will subsequently teach.

Mathematical Understanding and Preservice Teachers

An analysis of mathematical understanding from the perspective of the learner was first developed by Skemp (1976) who categorised it into two distinct types: relational understanding, the ability to deduce specific rules and procedures from more general mathematical relations, and instrumental understanding, the ability to apply a rule to the solution of a problem without necessarily understanding how it works (Long, Dunne, & Kock, 2014). Relatedly, a decade later, Hiebert and Lefevre (1986) articulated the difference between conceptual and procedural knowledge and understanding. Conceptual understanding is generally defined as the development of links between existing knowledge and new knowledge (Hibert & Lefevre, 1986) whereby these links provide a source of meaning to mathematical procedures (Eisenhart et al., 1993), that may be applied in diverse contexts (Bale, 2006). This conceptual understanding aligns closely with Skemp's (1978) relational understanding, sometimes referred to as knowing 'why'. Procedural understanding refers to the mastery of computational skills (Eisenhart et al., 1993), knowledge of the algorithms, procedures and mathematical strategies, the use of the correct language and symbols (Hibert & Lefevre, 1986). This understanding aligns with Skemp's instrumental understanding, knowing 'how' to complete a mathematical skill or operation, but not necessarily 'why' it works. For the purpose of this research these definitions of conceptual and procedural understanding will be applied.

Both procedural and conceptual understanding are considered necessary dimensions of mathematical understanding (Eisenhart et al., 1993). Usiskin (2012) further subdivides a comprehensive understanding of mathematics into five dimensions of understanding: skill algorithm (how to complete a mathematical procedure or algorithm e.g. multiplication of two fractions), property proof (knowing why, e.g. when we multiply two fractions, an understanding why we multiply numerators together and denominators together); use application (knowing when and where to apply this algorithm or skill); representation-metaphor (being able to accurately represent a concept visually), and history-culture (a knowledge of the development of a mathematical concept over time, or perhaps its use in modern culture). Usiskin stated that all dimensions, with the exception of history-culture are important in the teaching and assessment of mathematical learning. It is this framework of mathematical understanding which is used in this research. While Skemp (1978) stated that procedural and relational understanding were often distinct, Usiskin emphasized that the dimensions are interconnected and they should be developed as such, simultaneously.

The widespread reform of problem-based mathematics teaching and learning has increased demands on teachers, requiring an even deeper mathematical knowledge, which extends further than simply knowledge of the syllabus (Silverman & Thompson, 2008). Silverman and Thompson acknowledged the broad aim for all teachers of mathematics to develop 'deep personally powerful mathematical understandings' (2008, p. 507), which could be aligned with the concept of 'conceptual knowledge'. It is this conceptual knowledge that is needed by teachers to foster the classroom experiences and models, pose higher-order questions (Bloom, 1956) and diagnose student strategies and misconceptions (Holm & Kajander, 2012) however US research has indicated that many teachers lack the deep, specialized mathematical knowledge needed for effective teaching (Selling et al., 2016), and this is also reflected internationally (Hill & Ball, 2004; Ma, 2010; Tatto, 2013).

Ball (1990a) asserts that the mathematical understandings that primary level preservice teachers bring to the classroom are inadequate for teaching for understanding. Previous research on preservice teacher understanding have revealed a reliance on procedural understanding and exposed insufficient conceptual understanding(Ball, 1990b; Borko et al., 1992; Marchionda, 2006; Slattery & Fitzmaurice, 2014), with some participants indicating that they were relying on their mathematical understanding from school (Marchionda, 2006; Slattery & Fitzmaurice, 2014). Novotná and Hoch (2008) emphasise the danger of assuming prospective teachers know certain concepts just because they encountered them when they were in secondary school. The increased demand on mathematics teachers to foster problem-based learning requires teacher education programs to have accurate means of assessing this deeper level of mathematical knowledge for teaching (Selling et al., 2016). Eisenhart et al (1993) found that prospective teachers expressed teaching for conceptual understanding as a goal of their mathematical pedagogy, but felt the means to achieving this goal to be abstract rather than concrete.

Factorisation

Zhu and Simon (1987) define algebraic factorisation as the inverse of multiplication and outline the following procedure for factorising quadratics of quadratic coefficient 1, (i.e. quadratics in the form $x^2 + bx + c$). The procedure begins by finding all pairs of positive integers that factor the constant term *c* of the quadratic. The procedure is now dependent on the linear and constant terms. From the pairs of factors, select the pair whose algebraic sum equals the coefficient of the linear term, and assign the appropriate signs. If the constant term is positive, then the factors are both negative or both positive as the coefficient of the linear term is negative or positive, respectively. If the constant term is negative, then the two factors will have different signs (Zhu & Simon, 1987).

One procedure for factorising quadratics in the form $ax^2 + bx + c$ begins by first finding two numbers that multiply to give *ac*, by listing the factors of *ac*, and from these numbers choosing a pair that add to give b. The expression must now be rewritten to replace the middle term with this two numbers as coefficients of *x*. Now the procedure is essentially factorising by grouping in that the first two and last two terms should be factorised separately, to reveal a common factor. Others emphasise the use of an array model which promotes student understanding of representing multiplication and factorising as a portioning of numbers (Day & Hurrell, 2015).

There is general consensus that students find quadratic relations conceptually challenging (Kotsopoulos, 2007; Leong et al., 2010). As identified by Leong et al. (2010) the possible barriers to understanding include students' view of factorisation as completely abstract, a lack of sufficient algebraic skills and the conception of factorisation as a purely examinable skill without a broader context for which it may be used. To counteract the barriers to student understanding, Leong et al. (2010) suggest a new approach to the teaching of factorisation that would appear concrete and sensible to students and encourage a broader context of factorisation as reverse expansion. They found AlgeCards proficient in encouraging the concept of "factorisation as forming rectangle and finding length/breadth" (Leong et al., 2010, p. 21). AlgeCards, also called Algebra tiles, are mathematical manipulatives used to model integers and variables. These tiles are helpful in the modeling of multiplication and factorisation as their dimensions are based on the concept of area, thus providing a visual representation for students.

Hoch and Dreyfus (2004) supported the work of Linchevski and Livneh (1999) in emphasising the importance of structure sense in algebra. They present the writing of a quadratic expression as the product of two linear factors, identifying this as simply different interpretations of the same structure, and emphasise the importance of students seeing algebraic structure as an expression or equation before applying algebraic transformations (Hoch & Dreyfus, 2004). Kotsopoulos (2007) identified that the source of her students' difficulty lay in recognising and understanding varied representations of the same quadratic relationship. As the factorisation of quadratics requires students to be able to quickly find factors of one number that also add to find another, the way students, and therefore prospective mathematics teachers, learn multiplication facts impacts their conceptual understanding of factorisation (Kotsopoulos, 2007).

Factorising is a compulsory part of most mathematics curricula, and therefore is something almost all preservice teachers will go on to teach. It is a topic that is typically taught in the earlier years in secondary school and not necessarily taught in a formal sense during degree programmes, though it is a concept that is embedded within other concepts e.g. simplifying algebraic fractions. It would normally be assumed knowledge at degree level stage of mathematics education.

Methodology

The aim of this research is to assess preservice mathematics teachers' understanding of quadratic factorisation by answering two specific research questions:

1. What Procedural Understanding do preservice teachers have of quadratic factorisation?
2. What Conceptual Understanding do preservice teacher have of quadratic factorisation?

Ethical approval was granted from the local research ethics committee. Participants (n=83) were drawn from first to fourth year of an undergraduate programme in physical education and mathematics teacher education, and both first and second year of a two-year professional master's in education for trainee mathematics teachers at one Irish university. Both programs involve two (one six-week and one ten-week) school placements. Information sheets and consent forms were distributed and all potential participants were informed that participation was voluntary and, if they did decide to participate, they could withdraw at any later point, if they wished. The response rate was 91.2% and participants were given one hour to complete the questionnaire.

The researchers used a single instrument (appendix A) to assess preservice teacher understanding, and categorise the understanding as either conceptual or procedural. Zhu and Simon (1987) proposed several tests that when used together could assess understanding of quadratic factorisation. From those suggested, the tests of solving factorization problems, outlining a definition of factorising as the inverse of multiplication, and testing the verbal explanation of the procedures for factoring have been selected for use in our questionnaire. Our questionnaire is consciously aligned with Usiskin's Framework, which identifies several dimensions to conceptual understanding; skill algorithm, the property-proof, use-application and representation-metaphor (Usiskin, 2012), and is comprised of sixteen questions, each aligned with a dimension of understanding that when analysed together provide an insight into the procedural and conceptual understanding of the participants.

The undergraduate preservice teachers in this study complete 11 modules of mathematics and two modules of mathematics pedagogy over the four-year degree. The postgraduate preservice teachers have completed a level 8 degree, with mathematics as a major subject, prior to participation in this teacher education programme. In addition, on the programme they complete 2 modules of mathematics, one module of mathematics pedagogy, a short module on Mathematics Knowledge for Teaching, and a module on Statistics and Probability Knowledge for Teaching. Factorisation appears on the secondary school mathematics syllabus, however the preservice teachers in this study do not formally study factorisation within the mathematics modules in their teacher education programmes. It is considered assumed knowledge on entry to these programmes so is therefore not taught explicitly, but would appear within modules e.g. solving equations in algebra.

Instrument Design and Marking
Skill- Algorithm

Question four, five and ten require the participants to demonstrate their procedural understanding of factorising quadratic expressions. Question four is a quadratic expression with x^2 of coefficient 1, taken from a local secondary textbook. Question five is a quadratic expression with x^2 of coefficient 2, taken from the same textbook to assess participant ability to move beyond a simple trial and error method, and demonstrate their approach to factorising with negatives.

As proposed by Knuth et al. (2006) participant responses are analysed under the context of correctness. Responses are coded as correct, incorrect, no response or misconception. Responses that are coded as misinterpretation are those where the participant may have factorised the expression correctly, but mistakenly treated the expression as an equation and solved for x. Responses where the quadratic formula was used will also be coded as misconception.

Property- Proof

Question two, three and six require participants to provide definitions and explain their approach in their own words. The written and spoken vocabulary of a concept has often been seen to be indicative of depth of conceptual understanding (Usiskin, 2012). Kannemeyer (2005) adopted this approach to assess depth of understanding by examining how successful students are in presenting their explanations, and by examining the coherence of their explanations to the mathematical solution. For this reason, question two and three are included to provide an indication of preservice teacher fluency and understanding of the noun 'factors' and the verb 'factorise', while question six will assess their ability to provide a coherent explanation to a previously implemented procedure. In question seven the participants are required to demonstrate they have factorised the expression in question five correctly, which in essence is requiring them to prove their method of factorisation. The participants should expand the expression in order to display a conceptual approach to factorisation as the inverse of multiplication (Zhu & Simon, 1987).

Question two and three focus on the definitions of 'factors' and 'factorise'. Similar to the work of Knuth et al. (2006), responses here will be coded as procedural, conceptual, no response, inappropriate. For the purpose of this research, a conceptual definition of "factors" is proposed as any of the numbers or symbols in mathematics that when multiplied together form a product. A conceptual definition of "factorise" is one that relates to the concept of factorising as the inverse of the operation of multiplying (Zhu & Simon, 1987) or to find two or more values whose product equals the original value.

Question six responses are coded as relational (conceptual), operational (procedural) and no response (Knuth et al., 2006). Relational responses are those that refer to the procedure of factorising as the inverse of multiplication, and that are easily generalizable (Selling et al., 2016). Operational responses are those that focus on the numerical operations, with little focus on the appropriate mathematical terminology or reasoning behind the method.

Question seven, following the conceptual understanding of factorising as the inverse of multiplying (Zhu & Simon, 1987), a response is coded as correct only if the participant has expanded their previously factorised expression. Responses are therefore coded as correct, incorrect and no response.

Use- Application

Question nine requires the participants to provide a justification of studying factorisation that would be appropriate for their students. This prompts the participant to identify an everyday example relevant to the students' lives, or some justification within the study of mathematics, assessing their understanding of the application of quadratic factorisation.

A thematic analysis will be conducted to identify common themes amongst participant responses to question nine, the justification of factorising. Similar responses are grouped together.

Representation- Metaphor

There is a considerable body of research to suggest the benefit of multiple representations in developing mathematical thinking (Brenner et al., 1997; Pape & Tchoshanov, 2001). Question eight requires the participants to represent the process of factorisation of the expression $2x^2 - 7x - 15$ in any way they feel is appropriate. Representations such as the area model (see figure 1) are considered to convey relational (conceptual) understanding (Leong et al., 2010). Responses here are coded as appropriate, inappropriate or no response (Ball, 1990a).

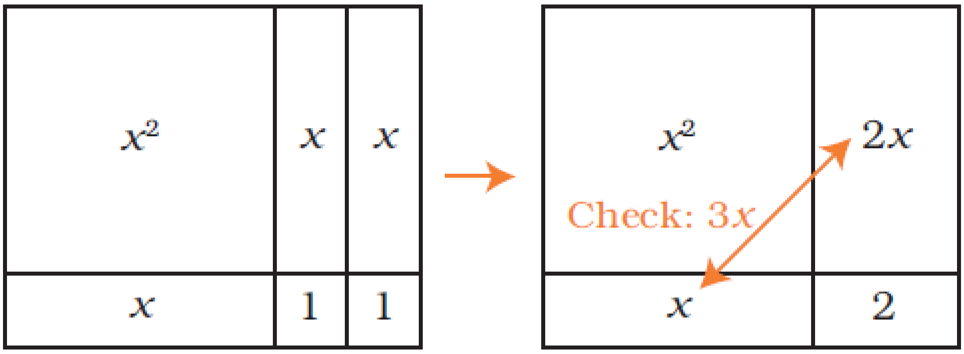

Figure 1: Hoong & Fwe (2010) Algebra tiles and rectangle diagram to represent factorising

Zhu and Simon (1987) suggest applying factorising to broader contexts, such as geometric representations, as a method of probing understanding. Question ten will assess participant understanding of the relationship of area to the problem, and ability to implement factorising as a solution method. Their ability to identify the correct solution through interpretation of the circumstances of the problem will provide an indication of both procedural and conceptual understanding. The responses to this question are coded as correct, if the response returns the dimensions of the frame as 3cm and 10cm, slip, if a numerical error is made and blunder, if a mistake indicative of misunderstanding is made.

Results

In this section the authors categorise the questionnaire responses of the 83 preservice secondary mathematics teachers who participated in this research, in a bid to address the research questions stated earlier. To provide a comprehensive assessment of mathematical understanding, this section has been categorised under the headings skill-algorithm, property-proof, use-application and representation-metaphor, the four dimensions of understanding as outlined by Usiskin's Framework (2012). Where relevant, examples of participant responses are included.

Skill-Algorithm:

Skill-Algorithm refers to procedural knowledge, knowing 'how' to perform a mathematical procedure, without necessarily understanding the underlying mechanics or reasoning behind it. Questions 4 and 5 were included to assess the participants' competency in this area. The results for questions four and five reveal high procedural knowledge with many participants factorising the expressions in both questions correctly (56.6% and 55.1% respectively). Responses that were factorised correctly, but the participant mistakenly solved for x were coded as 'misconception'. This occurred in 33.7% of responses to question four (table 1) and 39.8% of responses to question five (table 2). Responses where the students made use of the quadratic formula were also coded as misconception, with approximately 15.7% of participants doing so.

	Frequency	**Percent**
Correct	**54**	**65.1**
Incorrect	**1**	**1.2**
Misconception	**28**	**33.7**
Total	**83**	**100.0**

Table 1: Q4 Factorise $x^2 + 5x + 6$

	Frequency	**Percent**
Correct	**47**	**56.6**
Incorrect	**3**	**3.6**
Misconception	**33**	**39.8**
Total	**83**	**100.0**

Table 2: Q5 Factorise $2x^2 - 7x – 15$

The breakdown of responses by year of study are detailed in tables 3 and 4 below. The majority of First Year Undergraduate students mistakenly treated the expression as an equation in both question four and five (64.3%). Of note, 72.2% of First Year Postgraduate responses were coded as misconception in question five, with only 54.5% of responses coded as misconception in question four.

		First year	Second year	Third year	Fourth year	First year PME	Second year PME	Total
Correct	**Count % within year**	**5 35.7%**	**11 68.8%**	**12 80.0%**	**15 71.4%**	**5 45.5%**	**6 100.0%**	**54 65.1%**
Incorrect	**Count % within year**	**0 0.0%**	**0 0.0%**	**0 0.0%**	**1 4.8%**	**0 0.0%**	**0 0.0%**	**1 1.2%**
Misconception	**Count % within year**	**9 64.3%**	**5 31.3%**	**3 20.0%**	**5 23.8%**	**6 54.5%**	**0 0.0%**	**28 33.7%**
	Count % within year	**14 100.0%**	**16 100.0%**	**15 100.0%**	**21 100.0%**	**11 100.0%**	**6 100.0%**	**83 100.0%**

Table 3: Q4 Factorise $x^2 + 5x + 6$

		First year	Second year	Third year	Fourth year	First year PME	Second year PME	Total
Correct	**Count % within year**	**3 21.4%**	**12 75.0%**	**11 73.3%**	**13 61.9%**	**3 27.3%**	**5 83.3%**	**47 56.6%**
Incorrect	**Count % within year**	**2 14.3%**	**0 0.0%**	**0 0.0%**	**1 4.8%**	**0 0.0%**	**0 0.0%**	**3 3.6%**
Misconception	**Count % within year**	**9 64.3%**	**4 25.0%**	**4 26.7%**	**7 33.3%**	**8 72.7%**	**1 16.7%**	**33 39.8%**
	Count % within year	**14 100.0%**	**16 100.0%**	**15 100.0%**	**21 100.0%**	**11 100.0%**	**6 100.0%**	**83 100.0%**

Table 4: Q5 Factorise $2x^2 - 7x - 15$

Question ten was a word problem relating to area. The types and number of responses are displayed in Table 5 below.. The participants were largely successful in this question (80.7%). Responses coded as blunder (14.5%) were mainly cases where the participant struggled to interpret the information from the question and so could not form the correct equation.

	Frequency	Percent
No response	**2**	**2.4**
Incorrect	**67**	**80.7**
Slip	**2**	**2.4**
Blunder	**12**	**14.5**
Total	**83**	**100.0**

Table 5: Q10 Word Problem

Property-Proof:

The Property-Proof category refers to participants' relational understanding, their ability to explain the 'why' or the reasoning behind a procedure of operation, in this case factorising. For the questions selected for this purpose, the preservice teachers were required to explain what factors and factorising meant, and the mathematical justification for the process they carried out. In questions two and three a definition was coded as inappropriate if it did not convey the concept of factors as numbers that divide evenly into a larger number, or factorisation as the inverse of multiplication. Examples of this include "put the notation in its simplest form" and "factors are multiples of the numbers". The majority (45%) of definitions of 'factors' were conceptual in nature, for example:

'smaller divisors of a larger number"
"terms that divide equally into another expression"
"numbers that multiply together to give you that number"

55%, however provided procedural, inappropriate or blank responses (see Table 6).

	Frequency	**Percent**
No response	**6**	**7.2**
Conceptual	**45**	**54.2**
Procedural	**8**	**9.6**
Inappropriate	**24**	**28.9**
Total	**83**	**100.0**

Table 6: Q2 Define 'Factors'

In defining the verb 'factorise', the responses became more procedural in nature (35%) for example, '*what a term can be broken into'* and '*splitting it up'*.

	Frequency	**Percent**
No response	**1**	**1.2**
Conceptual	**13**	**15.7**
Procedural	**35**	**42.2**
Inappropriate	**34**	**41.0**
Total	**83**	**100.0**

Table 7: Q3 Define 'Factorise'

Question six required the participants to explain their procedure/approach to the factorisation of $2x^2 - 7x - 15$ in question five. Responses here were coded as relational, operational or no response. Only 22.9% provided a relational response that could be generalizable, for example

"*factors of the constant term multiplied by x^2 term coefficient, that add together to give back the x coefficient, but also when multiplied back give the constant value back correctly*".

69.9% of responses were considered operational, focusing on the procedure to the specific problem. Examples of this include:

"opened brackets and put in the x values, found factors of 15 that together with x values would give the middle number"
"rules that I learnt in school"
"I multiplied potentially correct factors in my head until I came up with the correct ones".

Approximately 7.2% did not provide a response.

Question seven required participants to demonstrate that they had factorised the expression $2x^2 - 7x - 15$ correctly. A response was only coded 'correct' if the participant expanded the factorised expression to return to the original expression, and 45.8% of participants did so. A total of 53% were incorrect. Cross-tabulation tests reveal that 81.8% of the participants who factorised $2x^2 - 7x - 15$ correctly, but mistakenly carried on to solve for x (misconception), also provided an incorrect response to the demonstration that they factorised the expression correctly (see table 8).

			Factorise $2x^2$ - 7x – 15			
			Correct	**Incorrect**	**Misconception**	**Total**
Demonstrate you have factorised Q5 correctly	**No response**	**Count % within 'Factorise $2x^2$ - 7x – 15'**	**0 0.0%**	**0 0.0%**	**1 3.0%**	**1 1.2%**
	Correct	**Count % within 'Factorise $2x^2$ - 7x - 15'**	**33 70.2%**	**0 0.0%**	**5 15.2%**	**38 45.8%**
	Incorrect	**Count % within 'Factorise $2x^2$ - 7x - 15'**	**14 29.8%**	**3 100.0%**	**27 81.8%**	**44 53.0%**
	Total	**Count % within 'Factorise $2x^2$ - 7x - 15'**	**47 100.0%**	**3 100.0%**	**33 100.0%**	**83 100.0%**

Table 8: Crosstabulation- Demonstrate you have factorised Q5 correctly * Factorise $2x^2$ - 7x – 15

Table 8 above shows how 29.8% of participants who were able to correctly factorise $2x^2 – 7x – 15$, were unable to demonstrate that they had factorised correctly.

Use-Application

The Use-Application questions assess participants' knowledge of where the skill/concept of factorising can be applied. For question nine, where participants were asked how they would justify the teaching of factorising to a group of students, a thematic analysis was conducted resulting in several predominant themes: no response (24%), solving equations (26%), other mathematical applications (30%), everyday example (10%) and area (7%) (see figure 2). The theme of mathematical applications was quite broad, however there were repeated responses of improving algebra skills (9.7%) and graphing (8.4%). The 10% everyday examples of the use of factorising were also quite vague and have been noted in table 9.

Everyday Examples
Building bridges/stadiums
Real life problems with some unknown
Animators- points on a plane
Modelling real-life situations-projectile motions
Engineering

Table 9: Everyday examples

Representation-Metaphor

The final category of the Usiskin framework refers to a person's ability to represent a mathematical concept in some pictorial or graphical way. It indicates an additional dimension of understanding if a person can represent a concept in addition to being able to complete and explain a procedure and state where it is applicable. For question eight only 10.8% of the research participants provided an appropriate representation of the factorised expression, with 44.6% providing an inappropriate response, and 44.6% not responding. From table 10 we can see that the majority (78.6%) of First Year Undergraduates did not attempt any representation. 81.8% of First Year Postgraduate students provided an inappropriate response of a graphical representation. This result may have been impacted by their examination of graphical representations of quadratic equations immediately prior to their completion of the questionnaire. Worryingly 66.7% of Fourth years did not provide a response, and almost 29% provided incorrect representations. This particular cohort of preservice teachers are mere months from qualifying as secondary school mathematics teachers.

		First year	Second year	Third year	Fourth year	First year PME	Second year PME	Total
No response	**Count** **% within year**	**11** **78.6%**	**3** **18.8%**	**6** **40.0%**	**14** **66.7%**	**2** **18.2%**	**1** **16.7%**	**37** **44.6%**
Appropriate	**Count** **% within year**	**1** **7.1%**	**2** **12.5%**	**3** **20.0%**	**1** **4.8%**	**0** **0.0%**	**2** **33.3%**	**9** **10.8%**
Inappropriate	**Count** **% within year**	**2** **14.3%**	**11** **68.8%**	**6** **40.0%**	**6** **28.6%**	**9** **81.8%**	**3** **50.0%**	**37** **44.6%**
	Count **% within year**	**14** **100.0%**	**16** **100.0%**	**15** **100.0%**	**21** **100.0%**	**11** **100.0%**	**6** **100.0%**	**83** **100.0%**

Table 10: Cross-tab tests on visual representation

Other relevant findings

Of the 83 participants of the study, only 13.3% have previously taught factorisation. Participant responses about the sources of their knowledge to complete their questionnaire are shown below in table 11, revealing that participants largely relied on their secondary school experience of the concept.

	Frequency	Percent
Secondary Education **Teacher Education program** **School Placement** **Other** **Total**	**72** **6** **3** **2** **83**	**86.7** **7.2** **3.6** **2.4** **100.0**

Table 11: Source of Understanding

When asked if they had ever been taught factorisation using any concrete or visual method, 19.3% said yes, 54.2% said no and 26.5% were unsure. Question fourteen questioned the participants as to whether they believe it would be of benefit to study the concept of factorisation during their teacher education degree, and the majority said yes (68%).

Discussion

In this section we discuss the findings of this study under the lens of the research questions.

What procedural understanding do preservice teachers have of quadratic factorisation?
Skill-Algorithm

The dimension of skill-algorithm directly refers to procedural understanding. The preservice teachers surveyed in this research displayed a strong procedural understanding of quadratic factorisation with only 1.2% and 3.6% answering question four and five respectively incorrect. An example of correctly implemented procedure is shown in figure 3. Barmby et al. (2007) acknowledge that often correct calculation of a problem reveals quite little about understanding, however a mistake may indicate the limitations of their understanding. The significance of this will be discussed later in this paper.

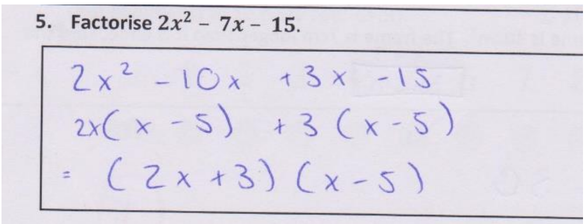

Figure 3: Example of correct procedure

The high number of successful responses to the word problem indicate that the preservice teachers had sufficient procedural understanding to accurately interpret the information from the problem, form an equation and factorise it appropriately to find an unknown value. There is often a focus in school and in examinations to interpret a word problem and form a mathematical equation around it (Lindvall & Ibarra, 1980; Nathan & Koedinger, 2000). This may explain the high number of correct responses to the word problem presented in question ten.

Property-Proof

In defining 'factors', only 9.6% of participants provided a procedural response, however many definitions of 'factorise' became significantly more procedural (42.2%), with some participants providing inappropriate responses that were mathematically incoherent, for example:

'to break up an equation'

Examples such as this may be founded on a procedural understanding, but have been poorly verbalised. Selling et al. (2016) suggested that explanations could be critiqued based on validity, generalizability and completeness. Operational responses did not provide a generalizable response that indicated why they were performing specific numerical operations, but simply listed steps. The written explanations of the process of factorisation were in the majority operational (69.9%), further indicating a strong procedural understanding of the concept.

Use- Application

The ability to provide an accurate suggestion of justification or use of factorising may indicate a deep level of understanding of the concept. Significantly, 26% of responses identify solving equations as a use or justification for factorising and this demonstrates a highly procedural understanding. Although some participants suggested everyday examples such as building bridges, budgeting, engineering and projectile motions, many responses were coded as vague. Examples of this include:

'identifying something common in an expression'
'breaking down an equation'
'for representing roots of an equation on a graph'.

There were many responses entirely focused on factorisation in relation to equations, as evidenced by the second and third example above.

Representation-Metaphor

Any representation of an area/array model was deemed an appropriate representation of the process of factorisation (Leong et al., 2010). Of the 44.6% of responses deemed inappropriate, many participants made use of a graph to represent the roots of the factorised expression (see figure 4). As will be discussed later, the decision to use a graph is based on the incorrect interpretation of $2x^2 - 7x - 15$ as an equation rather than an expression. This reflects a strong procedural understanding of how to factorise and what the roots of an equation visually represent, but is a procedurally incorrect response in the context of the questionnaire which focuses on the factorisation of an expression.

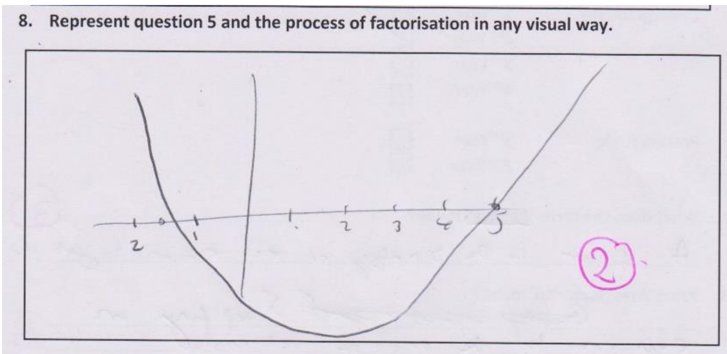

Figure 4: Inappropriate use of graph to represent factorised expression

In summary, across the four dimensions of understanding: skill-algorithm; property-proof; use-application and metaphor-representation, participants displayed a strong procedural understanding of the concept of quadratic factorisation. Questions involving factorisation procedure were implemented correctly and without any evident difficulty. Within representation-metaphor the use of a graph, together with the frequent responses to the use of factorisation provide a highly procedural perspective of factorisation as a method of solving quadratic equations.

What Conceptual Understanding do Preservice Teachers have of Quadratic Factorisation?
Skill-Algorithm

Although participants displayed a strong procedural understanding in their factorising performance, several participants treated the expressions in question four and five as equations and solved for x. This led to the use of misconception to code responses that were factorised correctly, demonstrating a strong procedural understanding, but were mistakenly solved for x. Misconception responses therefore accounted for 33.7% and 39.8% of responses to question four and five respectively.

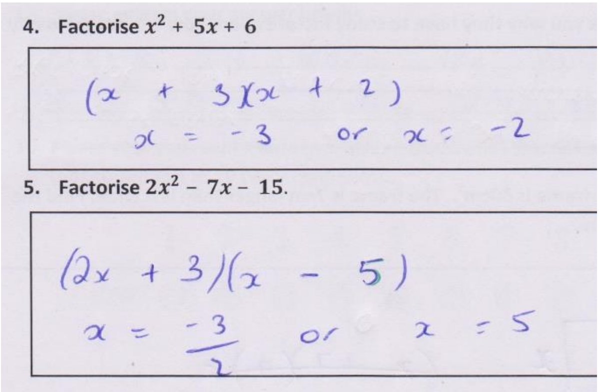

Figure 5: Example of "misconception" response

Powell (2012, p. 1) defined an expression as a combination of numbers and operations without an equal sign, while an equation is a 'mathematical statement where the equal sign is used to show equivalence between a number or expression on one side of the equal sign to the number or expression'. Students in school often misinterpret the equal sign (=) as an operational symbol, prompting them to find the answer, even though the equal sign should be viewed as a relational symbol (Sherman & Bisanz, 2009). With 86.7% of participants identifying that they were relying on their understanding from secondary school to complete the questionnaire, it is clear this misconception may have developed as students. The participants have not simply misinterpreted an equal sign, but have inserted an equal sign where one did not exist. They then begin the solving procedure accurately, but this procedure should not have been implemented to begin with. Usiskin (2012) considered blindly responding to the prompts of a problem as indicative of a lack of understanding. The responses coded as misconception indicate a blind action of trying to solve for x, rather than identifying that the question was simply asking to factorise an expression.

The high percentage of correct responses to question four suggest a procedural understanding of trial and error, an appropriate method for x^2 of coefficient 1. However, when the x^2 coefficient was greater than 1, as seen in question five, considerably more students struggled. The use of the quadratic formula by thirteen participants (15.7%) presents a considerable lack of conceptual understanding. The significance of this is that although directed to factorise, the participants are not in fact factorising, therefore procedurally incorrect, and are incorrectly assuming the existence of an equation (see figure 6).

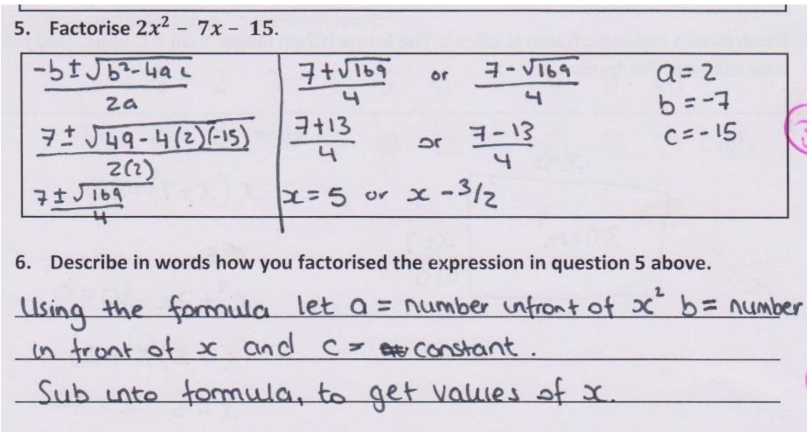

Figure 6: Example of misuse of the quadratic formula

Although factorising is often used as a method of solving equations, this quickness to solve an equation suggests a lack of conceptual understanding, and a procedural focus on ‘getting an answer’. Kieran (1981) suggests that the common misconceptions about the meaning of the equal sign might be the origin of students’ difficulties in dealing with polynomial expressions (Knuth et al., 2006). From the description of Thompson et al. (1994) a calculational orientation focuses on procedures as a means of getting answers. In this research, misconception responses were therefore evidence of a calculation rather than conceptual orientation, as the procedure of factorising was viewed only for finding “an answer” and not as the answer itself.

Property-Proof

Due to the mistreatment of the expression $2x^2 - 7x - 15$ as an equation, 45.8% provided an incorrect response by subbing in their values for x back into the ‘equation’ to demonstrate that it will equal zero, as can be seen in the example below (see figure 7).

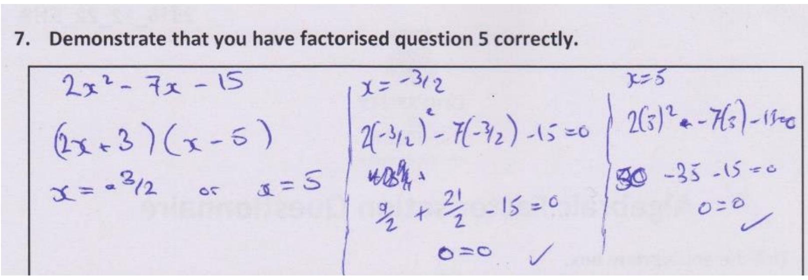

Figure 7: Inappropriate demonstration of correct factorisation

Usiskin proposed that something can only be understood if you identify the mathematical properties that underlie the method you have chosen, suggesting that in fact ‘understanding is contrasted by doing’ (2012, p. 6). Although 54.2% of the definitions of ‘factors’ were considered conceptual, many of these used terminologies which are not requisite of the definition, for example, ‘*breaking an equation up*’. A minority of responses made both procedurally and conceptually incorrect reference to *‘multiples’*. In explaining their approach to factorising the expression, 22.9% provided a relational response that could be generalizable, referring to the act of finding numbers that when multiplied together provide the correct term in the expression. This was suggestive of a conceptual understanding of the procedure they previously implemented.

Application- Use

Approximately 7% of the responses that indicated area as a potential use of quadratic factorising represent a conceptual understanding. Approximately 10% of participants attempted to provide a real-life example or justification for the study of factorisation, suggestive of an appreciation of the applications of the concept. The authors acknowledge however that these responses were quite vague and the precise function of factorisation within the application was usually unclear. Usiskin (2012) acknowledge that performance in the use-application dimension is generally lower than the performance in the skill-algorithm dimension, and attributes this to the little amount of time dedicated to teaching applications of a skill.

Representation-Metaphor

Only 10.8% of participants provided an appropriate representation of the factorisation of the expression, with many instead drawing a graph due to the interpretation of the expression as an equation. Although the area/array model is now commonly used for expanding brackets, few participants could identify that as factorisation is the inverse of multiplication, the array model could be used to find the factors of the expression. Any reference to an area/array model suggested conceptual understanding of factorisation as finding length and breadth when given area (Leong et al., 2010), further reinstating the concept of factorisation as the inverse of expansion (Zhu & Simon, 1987) (see figure 8). Only a minority (19.3%) of participants could recall being taught factorisation using some form of visual representation, therefore the requirement on participants to provide a visual representation as part of the questionnaire could be considered as a new context. Inability to apply mathematical understanding and logic in unfamiliar contexts is generally the result of a procedurally focused instruction in isolation of the conceptual meaning behind it (Hiebert & Grouws, 2007; Hiebert et al., 2005).

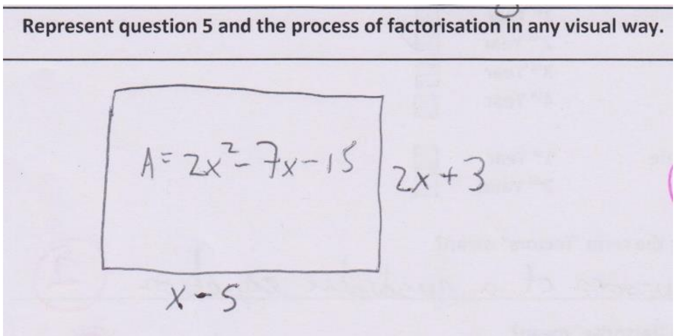

Figure 8: Appropriate visual representation of factorisation

Conclusion

In summary, responses provided for definitions and implementation of factorising procedure (skill-algorithm) indicate a majority conceptual understanding of the concept of factorisation. However, few appropriate responses to the visual representation of factorisation, a majority incorrect response rate to the demonstration of correct factorisation and few relational explanations of approach to factorising indicate a low level of conceptual understanding within the context of representation-metaphor and property-proof.

The results of this research indicate that prospective mathematics teachers have a strong procedural understanding of the concept of quadratic factorisation. However, although the relationship between procedural and conceptual knowledge is viewed as bidirectional, with mathematical competence depending on both (Rittle-Johnson, Schneider, & Star, 2015), there was evidence of a lower level of conceptual understanding across Usiskin's dimensions of understanding (2012). Skill-Algorithm, although by its nature procedurally focused, revealed some conceptual misunderstanding in the quickness to interpret an expression as an equation and solve for x, and considerable conceptual misunderstanding in the use of the quadratic formula. The results of this research therefore suggest that although prospective mathematics teachers have strong apparent procedural understanding of quadratic factorisation, it appears for many participants, their procedural understanding may not be connected to a greater conceptual understanding.

The suggestion of Ball (1990b) that prospective teachers are relying on their knowledge from school rather than their teacher preparation program was reinforced by this research where 86.7% were relying on their understanding from secondary school to complete the questionnaire. Bloom et al. (1956) and Holm and Kajander (2012) described the importance of conceptual knowledge for fostering significant classroom learning experiences and higher-order questions. Following the perspective of Hiebert (1999) that teacher understanding is the most significant factor in the reform of mathematics teaching, the procedural focus may be a cause for concern. The understanding of the teacher impacts the understanding of the student and so the need for conceptual understanding on the part of the teacher cannot be over-emphasised (Novotná & Hoch, 2008).

Eisenhart et al.(1993) suggest that prospective teachers express a desire for teaching for conceptual understanding. In this research study 68% of participants indicate that they would like to have examined the concept of factorisation during their teacher education program. In their explanation of their rationale for wanting to study factorisation more in-depth, 22% of participants describe an interest in improving understanding while 18% are eager to learn different approaches for teaching for understanding. These findings support the perspective of Eisenhart et al. (1993) and potentially indicate that these prospective mathematics teachers are not convinced of their conceptual understanding.

The literature emphasises the importance of both procedural and conceptual understanding of teachers if they are to teach for understanding, as is the aim of international reform. The findings of this research indicate that conceptual understanding should not be assumed, as misconceptions within some dimensions of understanding may lead to misconceptions in other dimensions. The authors do not suggest that procedural understanding should be ignored, but rather concepts should be considered in a way that links procedures to their conceptual meaning (Hiebert & Grouws, 2007). The assessment framework developed in this research may be used to assess either student or preservice teacher conceptual understanding of other mathematical concepts, allowing gaps in understanding to be addressed.

The data collection instrument used in this research provides a framework for assessing understanding that could be applied to any concept, at both secondary and tertiary level. Selling et al. (2016) argued that the growing demands on teacher mathematical knowledge for teaching require accurate methods of assessing this knowledge. The use of Usiskin's Framework (2012) to assess prospective teacher mathematical understanding as demonstrated in this research may be used to assess understanding of other concepts, so that misconceptions may be rectified during the teacher education program, and to ensure design of modules that address widespread misunderstanding.

While there is not sufficient time to cover all secondary school topics that preservice teachers will teach, it is important to demonstrate to them that understanding is multi-faceted.

Introducing them to Usiskin's framework of understanding within their pedagogy modules alerts these teachers to the fact that procedural understanding alone does not constitute sufficient understanding to teach mathematics in a comprehensive way.

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Appendix

Algebraic Factorisation Questionnaire
(Spaces for answers were provided when distributed)

1. Tick the appropriate box.

Undergraduate **1st Year** ☐
2nd Year ☐
3rd Year ☐
4th Year ☐
Postgraduate **1st Year** ☐
2nd Year ☐

2. What does the term "factors" mean?
3. What does 'factorise' mean?
4. Factorise $x^2 + 5x + 6$
5. Factorise $2x^2 - 7x - 15$.
6. Describe in words how you factorised the expression in question 5 above.
7. Prove that you have factorised question 5 correctly
8. Represent question 5 and the process of factorisation visually.
9. If a student were to ask you why they have to study factorisation, how would you justify it?
10. The area of a rectangle frame is 30cm^2. The frame is 7cm longer than it is wide. Find the dimensions of the frame.

11. Did you teach factorisation during School Placement 1 or 2?

Yes No

12. Where do you feel you gained the knowledge to fill in the questionnaire?

When you were in secondary school

Your Teacher Education Program

School Placement

Other

Please state: ____________________

13. Can you recall ever being taught factorisation with any concrete/visual method?

Yes No Unsure

14. Would you have found it of benefit to have examined the concept of factorisation during your teacher education degree?

Yes No Maybe

15. Please explain your answer briefly:

16. Please rate your level of confidence in teaching factorisation for understanding.

	1	2	3	4	5	6	7	8	9	10	
No confidence											**high confidence**