

1998

Research in mathematics education: A contemporary perspective

Alistair McIntosh (Ed.)
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Research in Mathematics Education

A
Contemporary
Perspective

Edited by
Alistair McIntosh
and
Nerida Ellerton

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Research in Mathematics Education: A Contemporary Perspective

Edited by

Alistair McIntosh and Nerida Ellerton

MASTEC
Edith Cowan University
Perth, Western Australia

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National Library of Australia
Cataloguing-in-Publication Data

Research in mathematics education: A contemporary
perspective/edited by Alistair McIntosh and Nerida Ellerton. Perth,
WA: MASTEC, 1998.

ISBN 0-7298-0407-0

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Preface

The twelve chapters in this book—all but two written by researchers in Australian universities—provide ample evidence of the impressive contributions currently being made by Australia to research in mathematics education.

The authors' fields of inquiry are diverse: they include discussion of the roles of language and imagery, problem posing and problem solving, students' beliefs and students' thinking, gambling and mental computation. Their methodologies are no less diverse, incorporating descriptions of both quantitative and qualitative research projects, including action research in classrooms, theoretical perspectives and the development of theoretical models, reviews of research, surveys, clinical interviews and descriptions of new research tools.

The book, which was initiated by Nerida Ellerton while Professor of Mathematics Education at Edith Cowan University, has been brought to publication by myself as Director of the Australian Institute for Research in Primary Mathematics Education, which sponsored the publication.

Particular thanks are due to our research assistant, Amanda Kendle, who has been involved at every stage of the production of the book and has carried out this work with a remarkable combination of efficiency, insight and unfailing cheerfulness.

Thanks are also due to my colleagues Jack Bana and Tony Herrington who proof-read the chapters and made many valuable suggestions.

We hope that the individual chapters will stimulate interest in particular topics and research perspectives, and that the book as a whole will provide a useful indication of the scope and possibilities of research in mathematics education at the present time.

Alistair McIntosh
Director, Australian Institute for Research in Primary Mathematics Education, Edith Cowan
University

July, 1998

Student Writing and Mathematics Learning

Mal Shield

The use of writing as an aid to mathematics learning has been the subject of a large number of teaching and research publications in the past fifteen years. Before examining a range of these studies, it is necessary to consider the background to these types of learning activities. While writing has always been an integral part of mathematics learning, it has usually been restricted to the symbolic recording and reproduction of algorithms and proofs, usually in a highly repetitious manner. In the research considered in this chapter, the term writing is used to describe deliberate tasks involving more extended forms which include prose as well as symbolic expressions and diagrams. The use of writing has been a focus of interest in teaching and learning in most subject areas. The earlier "writing across the curriculum" concept has more recently been replaced with the idea of "writing to learn." While the earlier approach focused on the conventional forms of writing in each particular field, writing to learn generally uses less formal writing with the aim of having learners use language to develop their understanding of the subject material (Connolly, 1989). Even in a field like mathematics with its well developed forms of exposition, the use of natural language is an important component of a writing to learn approach. The focus is on the benefits to learning derived from the process of writing rather than on the final written product.

Emig (1977) argued that the process of writing allowed students to engage in formulating, organising and evaluating ideas. Harley-James (1982) cited several reasons why writing benefits student learning. These included the focusing of thoughts and providing for more complex thought because the language is made visible, and assisting in the translation of mental images into language. Rose (1989) and others attribute a range of benefits for students from participating in learning through and about writing. These benefits include building on their own experiences, developing language abilities especially writing fluency, being active participants in the classroom, becoming personally engaged in the learning and facilitating communication with the teacher. The importance of developing communication in mathematics in all its forms has been highlighted in recent curriculum documents; for example both the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) and *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) have specific sections related to communication and the use of writing.

There is considerable agreement that understanding in mathematics should be thought of in terms of networks of internal representations (Hiebert & Carpenter, 1992) and that learning with understanding involves making meaningful connections among external and internal representations of new concepts and

representations of existing knowledge (Baroody & Ginsburg, 1986). Skemp (1976) had earlier described such understanding as "relational" and contrasted this with what he termed "instrumental" understanding which is characterised by a knowledge of many discrete mathematical processes which can be applied in very limited situations. Given the apparent benefits of a writing to learn approach listed above, it is not surprising that it has attracted the attention of mathematics educators seeking to move mathematics learning away from the instrumental "tell-show-do" approach (Baroody & Ginsburg, 1990).

The products of student writing may be placed in one of two main categories described by Britton, Burgess, Martin, McLeod and Rosen (1975) as "transactional" and "expressive." Transactional writing is the writing of a participant in a discourse with the purpose of informing, persuading or instructing. In the case of classroom writing, the audience is usually the teacher and such writing has been used extensively in summaries, essays, reports and assignments. Expressive writing is a more reflective process described by Oaks and Rose (1992) as "thinking aloud on paper" (p. 14). It allows the learner to consider personally the meaning and significance of current activities. One example is known as "freewriting" which allows the learners to generate thoughts and explore feelings for their own benefit with no real sense of an external audience. Expressive writing is usually associated with the use of a journal and may also occur in letter writing. It is often not possible to classify a piece of writing as purely transactional or expressive. It is more useful to consider a transactional-expressive continuum along which a text may be placed. Powell and Lopez (1989) argued that writing for the purpose of learning with understanding needs to be located somewhere on this continuum away from the purely transactional end.

Studies of Writing in Mathematics Learning

For this review, a selection from the large number of case studies on the use of writing activities in the learning of mathematics will be examined. These studies generally lack a coherent and systematic method of analysis but generally claim substantial benefits to student learning of mathematics. One of the difficulties of investigating the benefits of the use of writing to learn is that there are so many ways in which writing may be used and so many potential benefits which may be realised. Following the review of these studies, the work of four groups of researchers who have attempted more rigorous approaches to the study of the use of writing activities in mathematics will be discussed.

Small-Scale Case Studies

A large number of studies based on the use of writing activities in one or a small number of mathematics classes has been reported. These studies report a wide variety of approaches to the inclusion of writing activities. Many of the studies have simply accepted the general outcomes of writing to learn outlined above and assumed that similar outcomes are being realised in mathematics classes, without any basis for analysis of the writing to substantiate these claims.

Anecdotal evidence has often been used to support the efficacy of writing as a strategy for learning mathematics. As stated by Powell and Lopez (1989):

A number of mathematics educators have asserted that writing facilitates mathematics learning; however, little evidence of students' conceptual development or increased mathematical maturity has been proffered to support the reasonableness of this assertion. (p. 160)

While these small-scale case studies have not really demonstrated an increase in student understanding of mathematics as a result of the use of writing activities, they have, however, provided reasonable evidence of other benefits to learning including the increased dialogue between teacher and student and the exposure of student misconceptions which might otherwise have been hidden for a longer time. For this discussion, the selection of studies will be divided into two groups, those which focus predominantly on journal writing and those concerned with a variety of writing tasks aimed at eliciting mainly transactional responses. This division is based more on the stated focus of the study in question rather than the nature of the writing reported. While journal writing is often associated with the expressive end of the continuum, much of the writing reported to have been included in journals is purely transactional.

Journal Writing

There is some variation in the definition of journal writing apparent in the writing in mathematics learning literature. In general, journal writing requires a regular series of writings throughout a course and these are kept in some sort of notebook dedicated to the purpose. However, within this broad category, much variation is possible. Nahrgang and Petersen (1986) described a journal as a "diarylike series of writing assignments" (p. 461). McIntosh (1991) called such a set of tasks a learning log and differentiated this from a journal which she described as less formal and more communicative. The conception of a journal enunciated by Borasi and Rose (1989) was one in which "students can write down any thoughts related to their mathematics learning" (p. 348).

Nahrgang and Peterson (1986) made use of journal writing with their college mathematics classes. The writing tasks formed part of the students' assessment for the course. Journal entries were made by students in response to specific prompts such as: "Discuss the following statement: 'Factoring and finding a product are reverse processes'" (p. 463). Nahrgang and Peterson observed that this task encouraged students to link the idea of factorisation with a number of ideas from earlier learning such as multiplication and division, and simplifying expressions. They believed that such writing could contribute to both "the understanding of mathematics concepts and the ability to express that understanding" (p. 465).

Journal writing was used as part of a calculus course by Mett (1989). She recognised the need to instruct the students in what to write in their journals at the beginning of the course, providing guidelines and examples of journal entries. Students were instructed to consider three aspects in the writing: a summary of new material learned in class; a discussion of individual work outside class; and an analysis of connections, difficulties, and open questions. The journals were

collected weekly and contributed ten percent of the marks for the course grade. Other than the initial guidelines and examples, no specific prompts were given, students being expected to write as new material was introduced into the course. Mett was able to recognise the thought processes and the meaning making of her students in their writing. The examples provided demonstrated that students were using some expressive writing in the form of personal reflections rather than simply transactional, textbook-like expositions.

In working with college students, Paik and Norris (1984) and Powell and Lopez (1989) introduced journal writing into courses on business statistics and developmental mathematics respectively. Both groups provided the students with a set of questions to write about in their journals. These questions were clearly aimed at having students elaborate their knowledge of what they were learning in a transactional way. The following are the questions used by Paik and Norris.

- (1) What are the new terms I learned today? State the definitions.
- (2) What other things have I learned?
- (3) Why do we need such concepts?
- (4) What are the relationships among them?
- (5) What are the examples and counter-examples? (What it is and what it is not.)
- (6) Imagine real-life adaptations or applications relevant to the material. (p. 249)

Using a control group/experimental group design, Paik and Norris demonstrated a difference in achievement in favour of the experimental (journal writing) group although statistical significance was not achieved. Powell and Lopez reported the development in the way one selected student expressed his mathematical ideas, noting that towards the end of the semester the writing approached a textbook style.

McIntosh (1991) used her version of journal writing when working with junior high school students. Her instructions to students on journal writing included an invitation to write the following items.

- New words or new ideas or new formulas or new concepts you've learned
- Profound thoughts you've had
- Wonderings, musings, problems to solve
- Reflections on the class
- Questions—both answerable and unanswerable
- Writing ideas (p. 431)

Among other tasks, Kennedy (1985) used what he termed "logs" with his lower-secondary mathematics classes. These writing tasks were very similar to those classified as journal writing by other authors. In their logs, "the students are writing to themselves about what they're learning" (p. 59). Kennedy noted the need to develop a trusting environment with students so that they could convey their feelings and ideas without fear of ridicule. He made the claim that such

writing "opens a door to awareness and understanding not just of math, but of how they think and learn" (p. 61).

Transactional Writing

The writing tasks in the studies discussed in this section generally require the student writers to describe or explain mathematical ideas, although some studies also include a small element of expressive writing. Miller and her colleagues (Miller & England, 1989; Miller, 1990, 1991a, 1991b) have written extensively about the use of writing "prompts" in mathematics classes. In the form of questions, these elicit expository writing which is mainly of a transactional style although some of the prompts also seek to have the students express their feelings about the mathematics they are learning and about the class in general. Miller described four types of prompts, namely contextual, instructional, reflective and miscellaneous. Reflective prompts were further categorised as requiring analysis or clarification. The following examples of prompts were provided in Miller and England (1989).

contextual—Do you think that algebra is an important subject for you to study?

instructional—Tell me what you think the goals or purposes of today's class were.

reflective (analytical)—Remember when you learned how to _____? Imagine that you are writing a note to your best friend to explain how to do this. Write your note assuming that your friend really wants to know how to _____ and that he/she must rely on you and only you for an explanation.

reflective (clarify)—What would you identify that you have done which has helped you the most or contributed most to what success you have had so far in this class? Please explain as fully as possible.

miscellaneous—What is your favourite single digit number? (pp. 301-302)

The main study reported (Miller & England, 1989; Miller, 1990) involved three classes of algebra (year 9) students in the United States. As with many studies of this type, there was no attempt to measure changes in the students' thinking although the likelihood of this was noted. The main objective of this study was to investigate the use of writing as a channel of communication between students and their teacher. A considerable amount of anecdotal evidence for the effectiveness of this communication was presented.

Davison and Pearce (1988a) have also written widely about the use of writing in junior high school mathematics classes. They proposed five types of writing that might be considered for use in mathematics classes. These were: direct use (copying); linguistic translation (changing mathematical symbols into words); journals (summaries and explanations); applied use (problem writing); and creative use (project report). Although no evidence was provided, the authors made the reasonable claims that the use of such writing tasks would improve students' abilities to communicate mathematically and may assist in breaking down the image of mathematics as a rigid set of rules and procedures. From empirical studies involving interviews with teachers (Pearce & Davison, 1988) and surveys of the content of textbooks (Davison & Pearce, 1988b), they concluded that writing was little used in mathematics classes and that there was very little

encouragement provided in the textbooks for students to write, other than the usual symbolic manipulation.

Evans (1984) was a fifth grade teacher involved in the San Diego Area Writing Project. She reported on and provided examples of three types of writing tasks, explanations of how to do something, definition writing, and "troubleshooting." In this project, students answered standardised tests on arithmetic and geometry before and after working on units in which the writing tasks were a significant element. One class used the writing activities while a control class in the same school covered the same mathematical content without the writing activities. While all students made gains after taking the units, the gains made by the writing class were greater than those of a control class. Of particular note was the result that lower achieving students in the writing class made the largest gains. Gains were reported as changes in the test scores and no statistical comparisons were shown. Evans claimed that writing "gives us one more tool to help our less capable students grow and learn" (p. 835).

In a potentially useful union, one reported study (Venne, 1989) involved the collaboration of a mathematics teacher and an English teacher with a year 9 algebra class. Again the evidence provided was in the form of a few anecdotes. Some useful writing appeared to take place when students were asked to write about verbal interpretations of symbolic equations. In the main task reported from the study, students had to write what was called a "six-paragraph explanation of the solution" (p. 66) of a pair of simultaneous equations in two unknowns. For this task, students were provided with a model and one "typically good" response was quoted. It is difficult to reconcile the author's claim that such writing is somehow creative and conveys some idea of understanding with the quoted example. Clearly the task with its model required the students to simply verbalise the algorithm with no attempt to explain why any of the steps are valid or why they are needed. Such a restricted genre is likely to reinforce the view of mathematics already possessed by many students.

Other studies relying on anecdotal evidence have reported on interesting approaches to transactional writing tasks. Burns (1988) suggested ways to use writing in support of word problem solutions. Keith (1988) discussed the use of "short explorative writing assignments" (p. 714) which included, among others, summaries, visual image translation and synopsising tactics for solving a problem. She particularly noted the diagnostic value of such writing and stressed the need for positive teacher responses to such student writing. LeGere (1991) had students write a personal mathematics history at the commencement of her course and included other tasks such as writing about a problem students were experiencing difficulties with and writing to prompts like "so far in this class ..." Morgan (1992) reported on the creation of a magazine containing the reports of lower-level grade 9 students about the investigations they were pursuing in class. This format appeared to provide a sense of audience for the writers. Morgan noted that most of the students did not use diagrams or colour in their final reports, although these would have been appropriate. She attributed this to the expectations which students have about what is included in mathematical presentations.

Two Analyses of Journal Writing

Borasi and Rose (1989) recognised the fact that most of the evidence cited in studies of writing to learn mathematics up to that time had been anecdotal. They set out to develop a systematic method for the analysis of journal writing in the context of a college mathematics course, Algebra for Professional Programs, taught by them. The purpose of journal writing was explained to students at the start of the course and students were expected to write in their journals each night. The journals were collected by the teachers every second Friday and returned to the students the following Monday with personal written comments from one of the teachers. When several students expressed difficulty with knowing what to write about, thirty-six suggested ideas were supplied including: "Respond to a particular class topic; Reflect on math ideas or feelings about math; Describe your favourite math class" (p. 351). Students were sometimes also given specific writing topics for their journals during class time. At the conclusion of the course, students were asked to write in response to the following questions:

1. How has writing in your journal affected your learning of mathematics?
2. How do you feel about journal writing for this course?
3. What are the benefits of journal writing for mathematics classes?
4. How could journal writing be changed to be more effective? (p. 351)

Twenty-three complete sets of journals and post-course question responses were collected. Borasi and Rose (1989) established a framework for a content analysis of this large data source by reviewing the literature on writing to learn. This theoretical framework guided the "manual" search of the data and the interpretation of recurring patterns in the students' responses in a version of grounded theory methodology (Glaser & Strauss, 1967). The result of this analysis was what Borasi and Rose called a "taxonomy of potential benefits of journal writing" (p. 352) which is summarised below.

Potential benefits as the students write their journal: therapeutic value; increased learning of mathematical content; improvements in learning and problem-solving skills; reconceiving one's conception of mathematics.

Potential benefits as the teacher reads the journals: better evaluation and remediation of individual students; responses to feedback on the course; long-term instructional improvements.

Potential benefits as students and teacher dialogue in the journals: development of more individualised teaching; creation of a supportive class atmosphere.

As the taxonomy was built up, the continuing search of the data provided the necessary empirical evidence to support the categories and the claims made about them. Borasi and Rose (1989) argued that the value of this writing experience was dependent on how much students could be enticed to write expressively rather than simply reporting final forms of mathematics in a transactional way. They also remarked on the value of the teachers' responses but noted the considerable individual differences amongst the students and the teachers in their responses to journal writing.

The Borasi and Rose (1989) analysis provided clear evidence of the value of journal writing in the learning of mathematics, albeit in the limited context of one particular tertiary mathematics course. Moreover, this study provided researchers in the field with a detailed framework to use as a basis for the further investigation of journal writing in other mathematics learning contexts.

Waywood and his colleagues (Clarke, Waywood & Stephens, 1993; Waywood, 1992) devised an alternative approach to the analysis of student journal writing in mathematics. In an experiment involving approximately 500 students over a four-year period in one Victorian secondary school, these researchers examined, among other aspects of writing, the development of students' mathematical thinking and beliefs which might be fostered by the use of journal writing. In the school at the time of the investigation, journal writing contributed 30 percent of the assessment marks in mathematics.

A number of useful classifications were developed in this research. Waywood (1992) described the writing processes in which students might engage during journal writing as "summarising, collecting examples, questioning and discussing" (p. 37). Within each of these categories there were four sub-categories which enabled a detailed description of the processes evidenced in each student's writing. In conjunction with these four categories, a set of progress descriptors for assessing journals was developed. For example, for the category exemplification, two of the descriptors were:

... Able to use examples to show how a mathematical procedure is applied.

... Able to choose examples that summarise important aspects of a topic, idea, or application. These examples are fully annotated to show their relevance. (p. 39)

From the pattern established for a student using the progress descriptors, a global progress categorisation of the student's journal writing into one of three modes, namely recount, summary, or dialogue could be established. Parts of the definitions of these modes reported in Waywood (1992) are reproduced below.

RECOUNT When students are writing in this mode, they interpret the tasks in terms of concrete things to be done; to write a summary means *record*: ...

SUMMARY When students are writing in this mode they interpret the tasks as requiring involvement. The involvement is utilitarian. Describing gives way to *stating* and *organising* ... Journals show students trying to form an overview ...

DIALOGUE When writing in this mode, students see the task as requiring them to generate mathematics. ... Summaries are about *integrating*; questions are about *analysing* and *directing*; examples are *paradigms*; and discussing is about *formulating* arguments (p. 38)

With further analysis of the students' journals, and also analysis of questionnaires for students and teachers developed for the purpose, Clarke, Waywood and Stephens (1993) confirmed and elaborated the nature of these progress modes and formulated a detailed description of student thinking associated with each. They were able to establish a general trend that, as experience with this type of journal writing increased, the students' writing tended to progress through the modes.

These descriptions provide a useful basis for considering the types of mathematical activities which may help to move students from the recount mode through to the dialogue mode. The authors also noted the link between the students' writing and their perceptions of their learning of mathematics.

A Linguistic Approach

In a British study into the writing of reports of mathematical investigations, Morgan (1996) made use of the work of Halliday (1973) in a linguistic analysis of student writing. Morgan was interested in the difficulties that students have in writing, for assessment purposes, reports that their teacher-assessors would consider appropriate. Halliday's framework proposed three metafunctions of language, namely the ideational, interpersonal and textual functions.

The ideational function addresses the nature of mathematics and mathematical activity, including the role of people in its creation and use. Morgan (1996) noted that the use of symbols and typical manipulations of mathematical objects in texts provided a certain view of mathematics and that the presentation was often devoid of any human agency, the use of non-active verbs contributing to this perception. The interpersonal function addresses the roles and relationships of the author and reader, and how the two are constructed as individuals by the text. Mathematics textbooks usually are written with an authoritative tenor which is sometimes also evident in student writing. Morgan discussed the use of personal pronouns (*I* and *we*) which "may indicate the author's personal involvement with the activity portrayed in the text" as well as "implying that the reader is also actively involved in the doing of the mathematics" (p. 5). The use of words such as *consider*, *suppose* and *let* also serve to bring the reader into the discussion. Morgan noted that a student writer who adopted the authoritative style of a textbook may be considered arrogant by the assessor, and that student writers often made use of personal pronouns. The textual function considers the way the text is constructed. Mathematical text often uses logical reasoning in providing a progressive account of the development of an argument. It also includes the functions of the various parts of the text such as definitions and examples. Morgan noted that teacher-assessors of student texts may be influenced by the lack of logical structure in some of the writing.

A Text-Analysis Approach

In an ongoing study of student transactional writing in lower secondary school mathematics classes, this author (Shield, 1994) developed a scheme for the detailed analysis of the mathematical content of such writing. This is in contrast with the macro types of descriptions discussed and used by Borasi and Rose (1989) and Waywood (1992). The aim of this analysis was to better appreciate the thinking being expressed by the students in their writing to enable consideration to be given to advancing this thinking. Applications of the analysis have been reported in Shield and Swinson (1994) and Shield (1996).

As the writing under consideration was intended to explain a concept or procedure, the features of an explanation identified by Leinhardt (1987) in describing teachers' lessons were used as a starting point. The features used were:

1. Identification of the goal.
 2. Signal monitors indicating progress towards the goal.
 3. Examples of the case or instance.
 4. Demonstrations that include parallel representations, some levels of linkage of these representations, and identification of conditions of use and non-use.
 5. Legitimation of the new concept or procedure in terms of one or more of the following—known principles, cross-checks of representations, and compelling logic.
 6. Linkage of new concepts to old through identification of familiar, expanded, and new elements.
- (pp. 226-227)

These features were combined with the general construct of the "elaboration" of a concept or procedure to produce a set of descriptors for the parts of a written explanation. The term elaboration has been used, particularly in the study of reading processes, to describe the linking and integration of information being read. For example, Hamilton (1989) described elaboration as follows:

Elaboration can be defined as any enhancement of information that clarifies the relationship between information to be learned and related information, e.g., a learner's prior knowledge and experience, contiguously presented information. (p. 205)

A number of investigations (Hamilton, 1990; Mayer, 1980; Reder, 1980) have shown how elaborative processing by learners improves comprehension and retention of new material in a variety of contexts. It has also been demonstrated that elaborative processing during learning enhances the problem solving ability of the learner in the domain of that new knowledge, apparently because the richly elaborated knowledge base provides more options for the generation of ideas in the problem context.

Van Dormolen (1985) devised a set of descriptors in a discussion of mathematics textbooks and the way mathematical ideas are expressed in them. As part of his description, van Dormolen noted the existence of "general expressions that have to be learned as knowledge" (p. 146). These are usually statements of definitions, rules or procedures and are often signalled in the text in ways including the use of bold type or by enclosure in a box. Van Dormolen called these "kernels." Verbal and symbolic statements in the text are described in terms of their "aspect of mathematics" and "level of language." The following is a summary of the categories within these features.

Aspects of mathematics:

- (a) theoretical—theorems, definitions, generalisations;
- (b) algorithmic—explicit "how to do" methods;
- (c) logical—the way we are allowed to handle the theory;
- (d) methodological—heuristic "how to do rules"; and
- (e) communicative—conventions, how to name a diagram, write a proof.

Levels of language:

(a) exemplary—demonstrative, related to a specific example; and

(b) relative—generalised, not related to a specific example.

Within each level, the language may be procedural or descriptive.

In the study by Shield (1995), over 300 examples of transactional writing by year 8 students in four classes in two schools were examined over a two-year period. Although a variety of writing tasks were used, the writing examples all essentially consisted of the explanation of a concept or procedure. Some of the many examples of student writing reported in studies by other authors were also included in the analysis. The analysis involved an examination of the way students elaborated the main idea (kernel in van Dormolen's terms) which was expressed as a definition or general statement of the procedure. The features of an explanation, aspects of mathematics and levels of language were used to guide the analysis using grounded theory methodology (Glaser & Strauss, 1967). It was not expected that student writing would actually resemble a teacher's explanation or a textbook, but it could be expected to exhibit some of these features as well as other unexpected features.

The following discussion of the analysis uses the terminology outlined earlier. One of the most notable results of the analysis was the consistency of the writing styles of the students, both within this particular study and in the examples from other studies. The student writing usually centred on a specific symbolic demonstration of the procedure, generally accompanied by a verbal description of the procedure in exemplary language (related to that specific example). A kernel in relative (generalised) language was present in approximately half the writing examples. In particular writing tasks, the other elaborations present depended on the mathematical topic. For example, most students included a number line in their letters about adding directed numbers, although the links between this representation and the symbolic demonstration were often not made clear. A notable characteristic of most of the writing was the expression of a purely algorithmic aspect of mathematics with only very rare attempts to explain why a procedure was used or to justify any of the steps. A small number of examples included links between the new procedure and prior knowledge as in writing about highest common factor. Five of the twenty-six students in that class recalled the definition of a factor before demonstrating how to find a highest common factor.

In terms of the features of an explanation of a specific mathematical procedure or concept, the elaborations listed below were used by students in their writing. No order of presentation in a student writing example is implied by the order of the list. None of the examples examined contained all of these elaborations and some such as legitimisation occurred rarely.

Kernel	definition or general statement of the procedure
Goal statement	identification of the concept or procedure being explained
Demonstration	a worked example of the concept or procedure elaborated with: (a) symbolic representation; (b) verbal description; (c) diagrammatic representation; and (d) statement of convention
Legitimation	justification for the procedure or part of it using known principles
Link to prior knowledge	extensions of prior knowledge, reference to everyday experience
Practice exercises	set of questions to be answered by the reader by modelling on the demonstration

As a demonstration of the analysis, the following example of student writing is examined. The year 8 student was asked to write a letter to an absent friend to "explain all about highest common factors."

Dear Jessy,

Well I know you've been away for a few weeks and Mrs ----- gave me the job to explain about highest common factor and lowest common multiple. Well let me start with HCF. A factor is smaller than the number you start off with. The factors are numbers which can divide into the number evenly for e.g., $16 = 1, 2, 4, 8, 16$. Do you see. HCF is highest common factor, so you take two numbers 12 and 16 and you list the factors. When you've done that you look and see the highest factor both of them have otherwise you always have one.

e.g., $12 = 1, 2, 3, 4, 6, 12$

$16 = 1, 2, 4, 8, 16$

They both have 4 so your answer is 4. Get it.

In the analysis of this example, the conversational remark in the first half of the first sentence is ignored. The second half of this sentence expresses the goal of the presentation. The kernel is stated in the two sentences which precede the symbolic demonstration. Apart from "*12 and 16*" which doesn't really enter into the statement, the kernel is stated in relative language and expresses an algorithmic aspect of mathematics. The demonstration is also algorithmic and is accompanied by some verbal elaboration. Before the kernel there is a link with prior knowledge in the discussion of factors.

The example analysed above is more elaborate than many collected at the same level. The writer was able to express the procedure in general terms and demonstrate it with a worked example, as well as provide a link with the key prior

concept in case the reader did not remember it. This example illustrates the type of algorithmic writing with a lack of justification which is typical of much of the reported transactional writing.

This method of analysis of transactional writing in mathematics provides the possibility of assisting students to advance their mathematical thinking in a similar way to the analysis of journal writing by Clarke, Waywood and Stephens (1993). The purpose of the ongoing research of this author is to ascertain whether raising the awareness of teachers and students of the features of mathematical writing can assist in the development of student mathematical thinking. It is hypothesised that writing tasks in which students have to express mathematical ideas in terms of specific examples and in general terms, and in which they have to elaborate these ideas by linking them with prior knowledge (both mathematical and everyday) and with multiple representations, and then justify their statements, will lead to a deeper understanding of mathematics. However, it is already apparent that the processes involved are complex and that many aspects of the context of the learning including the students' and teachers' beliefs about mathematics and its learning play a part in determining the success of writing tasks.

Conclusion

Research into the use of writing in mathematics learning has progressed during the past ten years. From a mass of classroom studies at all levels of education relying on little more than the general outcomes from other writing areas and anecdotal evidence, research has moved towards the establishment of theoretical positions from which more definitive conclusions may result. The continuation of this trend is important if the potential value of writing as a tool for learning mathematics is to be realised.

Shepard (1993) addressed the issue of the role of writing in conceptual development from a theoretical standpoint. He adapted the work of a number of cognitive psychologists to describe learning in three phases, namely initial, intermediate and terminal. Within each phase, writing categories were described with indications of the types of writing tasks which would be appropriate. The following two examples come from the initial and intermediate phases.

Report: summarising account of description (from memory) of an event taken from direct experience—no inferences made beyond the information given.

Low-level analogic: true generalisations are made but organisation and/or relationships are not perceived. (p. 290)

These descriptions bear some resemblance to those developed by Waywood (1992). Shepard's aim was to provide teachers and researchers with guidance for designing writing assignments in mathematics which could assist in moving students from the rote memory of algorithms towards a more abstract and integrative type of thinking. In a reply to Shepard's article, Davis (1993) took issue with the underlying assumptions of this approach to learning through writing. The debate about theoretical positions will continue.

Thus far there had been little attention given in the research to the question of the effects of the learning context on the way students write about mathematics. In a study at the Grade 6 level, Ellerton (1988) had students write letters to imaginary absent friends about the mathematics studied during the previous three weeks. The content of the letters reflected the classroom experiences of the writers who came from four classes taught by four different teachers. For example, the letters from a class where the focus was on algorithmic detail reflected this approach while those from a class in which mathematics was integrated across the whole curriculum wrote more generally over a wide range of topics. In a study by this author (Shield, 1995), the writing of two classes of Grade 8 mathematics students was examined over a six-month period in which they were experiencing an elaborative approach to mathematics involving regular hands-on activities, drawing and writing. The students' beliefs about mathematics and its learning were also investigated. Over the period of the study there was no discernible change in their writing about mathematics which always centred on algorithms with little elaboration. This was also reflected in their beliefs about mathematics and its learning. It was concluded that in spite of the broader approach of their current teacher, their writing reflected their beliefs based on the previous seven years of mathematics instruction reinforced by textbooks and examinations. Perhaps any approach to the use of writing activities in mathematics needs to be part of a long-term change in the overall approach to the subject.

In Australia, there has been much development of the idea of a "genre approach" as a general strategy for the development of writing in various subject areas. In such an approach, learners are introduced to the various forms of writing used in particular fields and are made aware of the characteristics of such writing. The study by Venne (1989) discussed earlier is an example of such an approach, although the restricted nature of the genre provided in that study was of questionable value to student learning. Genre theorists, for example Kress (1982) and Christie (1993), argue that rather than constricting students' writing and thinking, the provision of a range of genre models in a field provides students with writing choices that they did not have previously. The students develop a greater facility to exercise linguistic choices which enables them to sometimes create new genres in response to their present contexts. The writing reported in most of the studies reviewed in this chapter was generally of a restricted form with some resemblance to typical mathematics textbook presentations so the introduction of new forms may be useful.

There are many opportunities for further research into the use of writing in the learning of mathematics. While research in this field has matured somewhat, there is still much scope for the development of theoretical positions which may form a basis for the fruitful application of writing in mathematics classrooms. There is a need to further consider the applicability of research on writing in other fields to writing in mathematics learning. The work of the genre theorists provides one possible line of further research. The identification and development of suitable genres is a major task in itself. Pimm (1987) posed a range of questions which are still current:

In conclusion, considerable attention needs to be paid to questions of how children record mathematics spontaneously, and what they find worthwhile to record in a particular context where both the purpose and need to record are clearly imposed by the constraints of the situation. In particular, what are the purposes to which disembodied language in the form of written records is put, and how might these purposes be conveyed to pupils? What do pupils find useful to record? Is the audience clear and known? Is the purpose known? What conventions are operating which govern the form in which the records should be written? These and many other questions seem to me to be central to an understanding of the place of writing in mathematics. (p. 137)

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Mathematics and Language

Ramakrishnan Menon

Many people believe that mathematics and language have very little in common, and might even insist that mathematics competency and language competency are negatively correlated. All too often, such beliefs arise from personal experiences (that is, friends who are “good at” mathematics but not at language and vice versa), and seldom from credible research studies. But now, with the ever-increasing emphasis on the importance of communication in mathematics, and the skills to access, select and interpret information, more and more studies on mathematics and language are being undertaken. In particular, there is a growing body of literature on writing to learn mathematics, arising out of the writing to learn movement. As well, numerous studies on the role of discourse (both oral and written) in the mathematics classroom have emerged recently, including the use of language to assess mathematics. In this chapter, I will draw both from theoretical considerations and research studies to explore the interrelationships between mathematics and language, as well as to suggest some implications for the classroom and for further research.

Language and Thought

Whether language shapes thought or thought shapes language has been a continuous debate. The Whorfian hypothesis (Whorf, 1956) suggests that language conditions thought, and that different cultures might have different concepts of, say, time and space. For example, because some American Indian languages like the Hopi Indian language, (and some Australian Aboriginal languages), do not consider time in a linear sense (e.g., do not express temporal ideas using spatial metaphors), Hopi mathematics would be quite different from the mathematics we are accustomed to.

According to Ellerton and Clements (1991), the implication of the strong form of the Whorfian hypothesis seems to be that “any attempt to teach mathematics to speakers of non-Western languages is a waste of time because their language has shaped their minds in ways that preclude the accommodation of Western ideas” (p. 23). While this strong version of the hypothesis is no longer accepted, a weaker version of the hypothesis has found favour.

This latter version draws on Vygotskian ideas to suggest that prior linguistic experience has to be accommodated or modified before concepts are crystallised and internalised. As well, while many mathematical concepts may be universal, some may not be (Bishop, 1988)—the concepts would depend on the language the learner uses to build those concepts and different world views might give rise to different, yet consistent mathematical systems.

Speaking from a slightly different perspective, Layzer (1989) states that "Whorf's argument assumes that natural science derives concepts like time and change from natural language. I think the reverse is true" (p. 126). He argues that although mathematics may begin by trying to make sense out of experiences, mathematics not only lends precision to natural language, but creates concepts that can be adequately expressed only in its own "unnatural—and hence universal—idiom" (p. 127). Even so, after creating these concepts, "we can talk about them in ordinary language" (p. 127).

Whether one agrees with the notion that language shapes thoughts or vice versa, there is no denying that language and thought are related. As Vygotsky (1962) says, "the child's intellectual growth is contingent on his mastering the social means of thought, that is, language" (p. 51).

But what are the links between mathematics and language, and how can language help or hinder mathematics learning? Before exploring some of these issues, let me discuss some links between writing and learning, as both the "writing to learn" (WTL) movement and the "writing to learn mathematics" (WTLM) movement are relatively recent phenomena, and researchers have shown increasing interest in these areas.

Learning and Writing

According to Emig (1983), writing is a unique mode of learning because it "connects the three major tenses of our experiences to make meaning" by "shuttling among past, present, and future" using the "processes of analysis and synthesis" (p. 129). She goes on to say that there is a marked correspondence between learning and writing, as both give self-provided feedback, generate and connect concepts, are integrative, active, personal, and self-rhythmed. Writing is also multirepresentational, in that it uses Bruner's enactive, iconic and symbolic modes of representation concurrently (or at least contiguously), with the hand, brain and eye all working in concert.

Writing has gone from an emphasis on a finished polished product to writing as a process-product, where drafts and redrafts are essential. The "whole language" movement (for example, Shanahan, 1991), which is currently in vogue, encourages learning through actively using language, rather than merely learning the rules of correct grammatical use.

Smith (1982), in discussing how language and writing help learning, says that writing helps us find out what we know and think because "language creates as well as communicates" (p. 67). Indeed, the very act of writing engenders thinking (Boyer, 1987; Elbow, 1981; Fulwiler, 1987; Luria & Yudovich, 1971; Zinsser, 1988).

In short, the proponents of writing to learn suggest that writing helps thinking in two ways: one, by the process of writing, one's thinking becomes clearer; and two, because it is a more permanent record, writing allows one to look more closely at, and revise what is written.

However, not all types of writing function to clarify and generate concepts. For example, Mayher, Lester and Pradl (1983) caution that "writing that involves

minimal language choices, such as filling-in-blanks exercises or answering questions with someone else's language—the textbook's or the teacher's—are of limited value in promoting writing or learning" (p. 78).

Writing to Learn Mathematics

Initially, it seems odd to associate writing, especially informal writing, with all its imprecision and hesitations, with the learning of mathematics, a subject considered precise, abstract and symbolic. This view is not surprising, given that "most children are very good at learning and using language—they make remarkable achievements in this domain before they commence schooling and in the absence of formal instruction—while very few children take so readily to mathematics" (Durkin, 1991, p. 4). Even so, the relatively recent phenomena of WTLM has spawned much research (Connolly, 1989; Ellerton & Clarkson, 1992; Menon, 1995a, 1995b, 1996; Porter & Masingila, 1995; Rose, 1989), ranging from students at the elementary school level to those at college level. For example, Porter and Masingila (1995) examined "the effect of WTLM on the conceptual understanding and procedural ability of students in an introductory college calculus course" (p. 325).

There is even a model of learning, the Adaptive Control of Thought (ACT) model, that links writing to mathematical problem solving (Kenyon, 1989). This model was a result of integrating ideas from computer science, information processing and linguistics (Anderson, 1983). In this model, learning is associated with two types of memory: *declarative* memory, where knowledge such as propositions, spatial images and temporal strings are stored; and *production* memory, where skills based on the construction and addition of procedures originating from other procedures and propositions are stored (Kenyon, 1989, p. 75).

According to Kenyon, there is a relationship between the ACT learning model, the three (recursive) phases of writing—namely the *prewriting* or planning phase, the *composition* phase and the *rewriting* phase—and the mathematical problem solving approach. When faced with a problem, a search is initially executed to access propositions from declarative memory and procedures from production memory. So, when one is at the prewriting or planning phase, one is attempting to understand what is being asked and what are the attendant conditions of the problem. A memory search (from both types of memory) for strategies and similar problems takes place at this initial phase of writing. Also, during this exploratory, prewriting phase, possible strategies are planned and a strategy is selected from both types of memory.

In the second phase—composition—the writing is more organised and cohesive in order to execute the strategy selected. This phase is analogous to the actual solving of the problem when a certain approach is implemented, for example, by using some known procedures. If the strategy selected does not lead to the desired solution, phase one is repeated, similar to the rejection of a procedure that did not lead to the desired solution and a search for another procedure.

In the third phase—rewriting—the writing resembles that of the transactional mode, as there is more clarity and focus because of the expected audience. This phase is similar to checking the reasonableness of the solution of a problem after which it may be either rewritten in order to communicate the solution to someone else or an attempt at a more elegant solution is made. Hence the ACT model of learning seems to link writing, learning and mathematics learning, albeit more specifically to mathematical problem solving.

Mildren (1992), too, suggests a link between writing and problem solving. He points out that just as learning language through expository essay writing involves "topic choice, planning and structuring text, organising information, drafting, revising, and editing" (p. 34), learning mathematics through problem solving involves "defining the unknown, determining what information one already knows, designing a strategy or plan for solving the problem, reaching a conclusion and then checking the results" (Bell & Bell, 1985, p. 212).

So far, some links between writing and problem solving in mathematics have been discussed. Other ideas and research results about WTLM will be briefly discussed later in the section on written discourse. Now I turn to some cautions about too uncritical an acceptance of WTLM.

The trivial type of mathematics that is reflected in some students' writing in mathematics (Caughey & Stephens, 1987; Ormell, 1992; Pengelly, 1990) is one area of concern. In addition, Pimm (1987) cautions that there are some disadvantages to using writing as a tool to learn mathematics. For instance, because of the effort required or spent in making the handwriting legible and clear, one may lose sight of what one was trying to express in the first place. As well, writing may interrupt our flow of thoughts, and it is usually very difficult to anticipate what assumptions need to be made explicit to the "uninformed" reader. Despite these concerns, there is sufficient empirical and theoretical evidence to support WTLM.

Mathematics Learning and Language Learning

Some Language Functions Common to Both Language and Mathematics

Laborde (1990) states that "the functions of language in the context of the mathematics classroom are those that have been recognised for a long time in the development of thought: Language serves both as a means of representation and as a means of communication" (p. 53). Some of the basic functions of language are to label, classify and compare and these are important both for making sense of the world and also in learning mathematics.

According to Sharron (1987), the classification of objects is a "complex operation which, depending on the sort of task, might require analytical perception, spontaneous comparative behaviour, the pursuit of logical evidence, systematic exploration, the conservation of constancies, and so on" (p. 68). Moreover, "it is through comparison that children move from the act of simple recognition of an object or event to establish relationships between them" (p. 63).

For example, to abstract the concept of a chair, some sort of classification into categories of things having the features of a chair have to be differentiated from

those that do not have the features of a chair. Then, in order to communicate to someone else that something is a chair, a label ("chair") has to be used. Also, when one differentiates a chair from a non-chair, one is comparing.

But the language functions of labelling, classifying, ordering, comparing and locating are not limited to language—they are common to mathematics as well. Indeed, in mathematics, such distancing and disembedding from the concrete is even more pronounced. For example, after using concrete referents such as the Dienes blocks for place value of numbers, one is expected to be able to manipulate the numbers, as if the numbers themselves were "concrete." Similarly, while seven apples can be seen and touched, the abstract concept of seven cannot be so seen and touched. Moreover, to understand "seven" more comprehensively, it has to be seen as, among other things, as one more than 6, one less than 8, etc.—that is, as a set of relationships and interlinked associations.

Generally speaking, while language represents tangible things, at least in the early stages, mathematics emphasises relationships and abstract ideas, even at the early stages: but both are used for a number of similar functions, such as classifying, labelling and so on. Moreover, just as language allows us to distance ourselves from, and extend, our experiences, concrete or otherwise, and to communicate to others, so too, can we use mathematical language to explain and extend mathematical concepts.

Linking Language Learning to Mathematics Learning

While "mathematics is not a natural language in the sense that English and Japanese are" (Pimm, 1987, p. 207), and learning language through immersion in language is different from learning mathematics through an artificial and temporary immersion into school mathematics (McIntosh, 1988), nevertheless language learning ideas can help mathematics learning (Menon, 1995c). For example, in language learning, a child moves from holophrases (such as "Mummy candy") to complete sentences ("Mummy, please give me some candy") through an interlanguage (Corder, 1981), such as "Mummy give candy." The interlanguage is not regarded as an error, but a necessary intermediate stage the language learner goes through, before attaining language competence. Similarly, in mathematics learning, a child can move through (possibly imprecise) everyday language (such as the "top" number of a fraction) to appropriate mathematics terminology (such as the "numerator" of a fraction).

In addition, the skills of listening, reading, speaking and writing are emphasised in language learning. According to Capps and Pickreign (1993), such skills should also be emphasised in mathematics learning. They suggest that mathematics instructional time should include exposure to, and reinforcement of, new mathematics terminology by listening to, as well as speaking, writing and reading the new words. Research also shows that specific instruction in mathematical terminology can clarify mathematical concepts (Garbe, 1985; Nicholson, 1989).

According to Del Campo and Clements (1987), listening and reading are *receptive* skills, while speaking and writing (as well as drawing, performing and

imagining) are *expressive* skills. Mathematics teachers tend to neglect the expressive skills, which "are more active, such as drawing triangles or explaining why the triangle is isosceles," and tend to overemphasise receptive skills, which "are more passive, such as identifying given figures as quadrilaterals or triangles" (Menon, 1995c, p. 1). Del Campo and Clements (1987) suggest that expressive tasks help children understand and remember mathematics concepts better.

Just as the teaching of communicative competence is associated with communicating effectively in context rather than merely learning grammatical rules and practising repetitive structural patterns, so, too, has context been shown to be important to meaningful mathematical learning (Capps & Pickreign, 1993; Menon, 1995a). For example, Menon (1995a, 1996) and Silverman, Winograd and Strohauser (1992) cite instances of students able to construct and solve meaningful word problems—an expressive task—situated in their own experiential context.

Even when learning mathematics with the help of computers, minor changes in the problem-solving context show gender differences in performance (Light, cited in Munro, 1992). For example, a king searching for his crown proved a more conducive context for boys than that of a picnic context, which proved more successful with girls.

The current approach to language learning is based on communicative competence (for example, Widdowson, 1990), rather than on the memorisation of rules. Similarly, it is widely accepted nowadays that communicating mathematical ideas is an important component of mathematical competence (National Council of Teachers of Mathematics, 1989). In particular, Borasi and Agor (1990) argue that many of the approaches to second language learning might be usefully modified to mathematics learning. For instance, by having students *interpret* rather than actually *plot* and *draw* a graph, students are using mathematics meaningfully rather than concentrating on the mechanics of plotting accurately—similar to using language in context, rather than just learning the rules of grammar.

Bilingualism and Mathematics

Research on bilingualism and mathematics has many issues to consider. First of all, while bilingualism is usually taken to mean that a certain level of proficiency is attained in both the first language, L1, and the second language, L2, in practice there are many "bilingual" programmes where the L1 proficiency is satisfactory but that of L2 is very low. Secondly, there are difficulties associated with understanding the language (L2) the teacher uses to teach mathematics and also that of the word problem itself. Moreover, the dissimilarities between the mathematics register and everyday (L1) language is compounded for bilinguals. As well, the L1 of the student may have no equivalent of a mathematical term or concept used in L2, which may be the language of instruction.

To explain the interaction between language and cognitive development in bilinguals, Cummins (1981) came up with a theoretical framework and proposed two hypotheses (Cummins, 1979): the first suggests two thresholds of linguistic competence—the *higher* threshold for those whose high competence in both languages will benefit cognitive growth and the *lower* threshold for those with low competence in both languages will impede cognitive progress. His second

hypothesis, that of *developmental interdependence*, proposes that cognitive development is affected by competence in both L1 and L2, and that L2 competence is dependent on L1 competence. In other words, if L1 is encouraged and valued, then L1, L2 and cognitive development are enhanced.

While research on bilingualism is not completely unequivocal about the effects of bilingualism on the mathematics competency of bilinguals, there is some research evidence attesting to the positive effects of bilingualism (Dawe, 1983). For example, Flores (1995) indicates that children's higher-order thinking skills were enhanced when instructions and discussions were in the children's L1, Spanish. Such research results indicate that learning more than one language facilitates abstract mathematical reasoning, given that a certain level of proficiency is attained in both the first language, L1, and the second language, L2 (Clarkson, 1991; Secada, 1988; Zepp, 1989). For a more comprehensive treatment of research in bilingualism and mathematics, please refer to Brodie (1989), and Ellerton and Clements (1991).

Difficulties Associated with the Mathematics Register

Just as different forms of language are used in different contexts (i.e., informal language used among close friends compared to the formal language used in an inauguration speech), the mathematics register is employed by those involved in mathematics to convey specific and appropriate meanings in mathematics. Unfortunately, to the uninitiated, the mathematics register confuses more than it enlightens. For example, Otterburn and Nicholson (1976) found that children were unable to explain various mathematical terms that were routinely used by teachers. Six ways whereby such difficulties arise are described next.

Register Confusion

Durkin and Shire (1991) show that ambiguity arising out of the use of similar words in different contexts hinders the learning of mathematical concepts, even though such misinterpretations diminish (but not completely disappear) with age. For example, words such as "odd, real, and right" have different meanings in everyday contexts compared to mathematical contexts. The book "Speaking Mathematically," by Pimm (1987), is replete with examples where students fail to distinguish between the registers of ordinary and mathematical English (for example, a right-angled triangle and a left-angled triangle).

Another, related, difficulty is the inconsistent way in which mathematics is communicated to the learner. As Carter, Frobisher and Roper (1994) state, "The language that teachers use to assist pupils in their learning of mathematics is frequently a barrier to that learning" (p. 125). For example, students tend to use the teacher-stated rule that area is "length times breadth" for a rectangle, to the area of a triangle. As well, counting objects "by touching one object after another and matching the touch with a number word" (p. 125) causes confusion as "ordinality, that is the order of touching the objects, is used to determine the cardinality of the number of objects so touched" (p. 126). That is, even though only one object is touched when saying "one," "two," and so on, each object, though one in quantity

is given a different cardinal name. As another example, Hanselman (1997) believes that terms such as reduce, cancel, invert and multiply can confuse the student and "should be treated like foul language and banned from premature use in the mathematics classroom" (p. 154).

It is also the case that teachers commit the sin of omission rather than that of commission when they assume the teacher-used mathematical terminology conveys the same meaning to the learner as to the teacher. For example, Otterburn and Nicholson (1976) found that children were unable to explain various mathematical terms that were routinely used by teachers.

Over-Extended Metaphors

Walkerdine (1988) believes that metaphors allow one to relate to familiar "discursive practice," and enable one to conceptualise something through the mental imagery brought about by the metaphor, rather than physically manipulating concrete objects. For example, the fraction $\frac{4}{3}$ could be thought of as "four pizzas divided equally among three people," without actually having to manipulate pizzas. Pimm (1987), too, states that metaphors both construct and extend meaning in mathematics. He distinguishes between two types of metaphors—the *extra-mathematical metaphor*, which explains mathematical concepts in terms of everyday objects and experiences (e.g., a graph is a picture, an equation is a balance); and the *structural metaphor*, which involves "a metaphoric extension of ideas from within mathematics itself" (p. 95), such as a spherical triangle and the slope of a curve. Because metaphors evoke powerful images by alluding to certain commonalities, they also tend to mask differences, making it all too easy to assign the metaphor with all (or most of) the characteristics of the original, literal meaning. Hence the problem of "overgeneralising" or in this case, "extending" the metaphor inappropriately.

For example, the expression "spherical triangle" is not a sphere-shaped triangle, as would be expected from the ordinary, adjectival use of "spherical" in "spherical container." Pimm gives another example of a metaphor—that "a complex number is a vector" (p. 105)—that might distort meaning. He says that while it is helpful to think of a two-dimensional plane to represent addition and subtraction of complex numbers, it tends to distort the concept of complex numbers when used for multiplication of complex numbers, as a complex line would be more appropriate here. Similar examples are cited by Gibbs and Orton (1994) who state that the metaphor "an equation is a balance" breaks down for quadratic equations, but is appropriate for linear equations; and by Kuchemann (1981), who states that the metaphor "algebra is a shorthand" may lead pupils to think that letters represent objects, rather than numbers, especially if, for example, the letter "a" is used for apples and the letter "b" for bananas often enough by the teacher.

Structural Differences

Other than lexical ambiguities associated with the difficulties in transferring from the everyday register to the mathematical register, the linguistic structure of everyday language can pose problems. For instance, Laborde (1990) notes that

"linguistic features of natural language can affect the transition of a situation from natural language into an algebraic statement" (p. 61). Researchers have shown that many tertiary students cannot translate relationships expressed in everyday language into corresponding mathematical expressions (Clement, Lochhead & Monk, 1981; Mestre & Lochhead, 1983). Translating word statements to algebraic equivalents has been noted to be especially problematic.

For example, students were asked this (classic) problem: "There are six times as many students as professors. If S represents students and P represents professors, write an equation connecting S and P." Most students translate it as $6S = P$, rather than $S = 6P$, as a solution to this problem. One explanation for this error is that the linguistic structure of the problem statement, where the expression "six times as many students" precedes the word "professors," could have influenced students into following the left-to-right order of the "everyday" sentence.

Levels of Language

Another factor associated with the transfer from the everyday to the mathematical register is the difference in levels of language required as one progresses from the colloquial to the mathematical (Freudenthal, 1978, pp. 233-242). New concepts are exemplified by *exemplary* or *demonstrative language* first. Later refinement leads to *relative language* and finally to *functional language*. Examples of these levels of language are as follows:

- a. *Demonstrative* (pointing out instances, without explanations): Half is like this part here.
- b. *Relative* (using words to indicate relationships or procedures): When something is cut into two equal parts, each part is called a half.
- c. *Functional* (generalisations or relationships between relationships): The common fraction $\frac{1}{2}$ is the same as the decimal fraction 0.5 because one out of two equal parts is equivalent to five out of ten equal parts.

The levels suggested by Freudenthal form a continuum ranging from personal referents to more abstract ones. In other words, according to Freudenthal, one first uses referents from one's own experiences (for instance, by pointing to or showing an example of a rectangle, when asked "What is a rectangle?") before moving on to using referents or abstractions that are further removed from idiosyncratic experiences (for instance, by stating the necessary and sufficient conditions for a quadrilateral to be a rectangle). While the levels of language might be a good way to analyse the language used in explanations of mathematical ideas, it would be wrong to assume that these levels of language progressively reach higher levels as one understands the concepts better. As Freudenthal (1978) himself explains, "most of us understand more language than we can speak" (p. 234).

According to Feuerstein—whose work was mainly with socially-handicapped underachievers—some children do not bother to express themselves clearly because they assume everyone else automatically understands what they are thinking (cited in Sharron, 1987, p. 65). This assumption is called "egocentric communication," and interestingly enough, was demonstrated in 1984, at a

conference in Oxford University on Feuerstein's work, by a group of psychologists and educational administrators who were asked to give a set of clear unambiguous directions, which, if followed correctly, would result in the drawing of a completed geometric pattern. Not one person in this group succeeded completely in communicating to a partner the correct way to draw the pattern! Indeed, "the majority found that the instructions, when they were comprehensible at all, resulted in anything but the required figure" (Sharron, 1987, p. 66).

Furthermore, Feuerstein (cited in Sharron, 1987) has found that, contrary to popular belief in the power of manipulatives to enhance mathematical concepts, "the extra emphasis in special education on physical manipulation of objects to aid learning is an extra obstacle—motoric acts frequently get in the way of children's attempts to formulate strategies abstractly" (p. 66) and by "simply inhibiting gestures like pointing or touching, which children with bad spatial orientation prefer because it is easier than thinking more abstractly about space, great improvements can occur" (p. 60). The implication seems to be that Freudenthal's demonstrative level of language might actually inhibit children from building up an internal reference system.

Linguistic Form and Semantic Structure of Word Problems

Another area extensively studied is the difficulties students have with word problems. It has been shown that the level of difficulty of a word problem is a function of not only the mathematical content of the problem, but also of "its linguistic form and semantic structure" (Gibbs & Orton, 1994, p. 102). Researchers from various parts of the world have consistently shown that students have great difficulties in understanding a problem because of the language involved, and not just because of the mathematics content of the problem (Clarkson, 1991; Clements & Ellerton, 1993, 1995; Lean, Clements, & Del Campo, 1990; Marinas & Clements, 1990; Newman, 1977). For example, consider the following two word problems:

Problem 1: Leo has 3 cookies. Mei Lee gave him 4 more cookies. How many cookies does Leo have now?

Problem 2: Leo has 3 cookies. Mei Lee has 4 more cookies than Leo. How many cookies does Mei Lee have?

Problem 1 is conceptually easier than Problem 2, although both use the word "more" and the numbers 3 and 4, and both give rise to the equation $3 + 4 = 7$.

One of the most influential studies in this area was that of Newman (1977). Briefly, the Newman study involved individually interviewing students who initially had their answers wrong to certain word problems, asking them certain specific questions as they worked through the problems they had got wrong earlier, and, on the basis of the verbal answers provided by the students, to classify their errors according to a hierarchy (errors in reading, comprehension, transformation, process skills, and encoding).

Newman (1977, 1983) has drawn attention to not only the influence of language factors on mathematics learning, but also to the inappropriateness of many assessment procedures and remediation programmes for mathematics in schools (Ellerton & Clements, 1992). Orr's (1987) work, which documents black children's

difficulties with mathematics and science due to differences in Black English and the standard English, further supports the notion that language factors play a very important part in the understanding of word problems.

It has also been shown that in certain languages, "the structure of the counting words reinforces the place-value of the numbers in a logical and consistent way," (Gibbs & Orton, 1994, p. 103). For example, the Chinese equivalent of the English "teen numbers" is "ten and three, ten and four," and so on, for "thirteen, fourteen," etc., and Fuson and Kwon (cited in Durkin & Shire, 1991) conclude that children using such a consistent place-value system are more facile at addition and subtraction. Adetula (1990), too, has shown that children perform better on word problems that are presented in the first than in the second language.

The "Cognitively Guided Instruction" programme by mathematics educators at the University of Wisconsin (Fennema, Carpenter & Peterson, 1989), recognises the importance of the difficulties students have with the semantic structure of word problems. Then, by making teachers acutely aware of these difficulties, the programme has succeeded in teachers modifying their teaching approach, resulting in their students making significant improvement in understanding word problems.

Language Used in Mathematics Tests

The role of language in testing has been explored by some researchers (see, for example, Davis, 1991). Results from such research indicate that children generally interpret the test tasks contextually. For example, if children interpret it as something for which the teacher requires an answer, they will give an answer, however nonsensical the question. Examples of questions are "which is heavier, red or yellow?" (Hughes & Grieve, cited in Davis, 1991), and "If there are 60 adults and 10 children as passengers in a bus, what is the age of the bus driver?" (Menon, ongoing research). It has been shown (Donaldson; Light; Pratt; Samuel & Bryant, all cited in Davis, 1991) that children who are supposed to be non-conservers on Piaget's well-known conservation tasks, can conserve if the experimenter modifies

- a) the question,
- b) the way it is put to the child (for example by not repeating the question), or
- c) the task itself (for example, by making the transformation accidental or incidental, rather than being the main focus of the task). Davis (cited in Davis, 1991) manipulated the phrasing of the instructions on a number of tasks given to five-year-olds, and found that children's answers varied according to the phrasing, and that the task was seen as requiring a "mathematical" response only if explicit mathematical terminology was used.

Discourse in the Mathematics Classroom

On the importance of discourse in the classroom, Pimm (1991) says that "externalising thought through spoken or written language can provide greater access to one's own (as well as for others) thoughts, thus aiding the crucial process of reflection, without which learning rarely takes place" (p. 23). Menon (1995b), for

example, found that oral discourse in the mathematics classroom could help the motivation and mathematical understanding of underachieving elementary school level children. According to Hicks (1995), discourse mediates learning, because, through meaningful classroom discourse, children realise what counts as legitimate knowledge in that discipline. Boucher (1998), Kazemi (1998) and Lampert (1988, 1990) have also documented children being empowered to do mathematics through such meaningful classroom discourse.

Numerous mathematics educators have focused on discourse in the mathematics classroom, and have concluded that when mathematics is taught for understanding as opposed to teaching a set of procedures, what constitutes legitimate mathematics knowledge changes from the external authority of the text or teacher to the internal authority of the learner. Indeed, there is sufficient research evidence to indicate that meaningful discourse improves performance on standardised mathematics tests (Hiebert & Wearne, 1993), as well as enhances understanding of mathematical concepts (Ball, 1991; Yackel, Cobb, Wood, Wheatley & Merkel, 1990).

Research on oral discourse in the mathematics classroom shows that meaningful discussion can take place only on the basis of shared assumptions and mutual understanding, or as Richards (1991) calls it, on the basis of consensual domain. Pirie and Schwarzenberger (1988) state that even non-mathematical talk does not necessarily inhibit meaningful mathematical discussion. Other researchers have examined group discussions (Perret-Clermont, cited in Cazden, 1988), discussion between pairs of students (Yackel, Cobb & Wood, 1991), teacher-led and student-centred whole-class discussions (Adams & Price, 1995; Miller, 1993), and genres of classroom discourse (Mousley & Marks, 1991), including the use of language demonstrating the asymmetric power relationship between teacher and students (Pimm, 1987).

According to such research, discussion allows for reformulation, precision, clarification, justification and ownership of ideas. For example, defining a mathematical term imprecisely or ambiguously to someone can give rise to examples contrary to what is expected by the one who gives the definition, and the need to justify oneself during explanations to others can give rise to queries for clarification or for the examination of unstated assumptions.

In spite of the many advantages of oral discourse in the classroom, Pirie (1991) warns of some disadvantages as well. For example, one's flow of thought might be interrupted, either when listening or when pausing to elaborate to others. Also, group discussions are only effective if a number of factors affecting group dynamics are taken into consideration. For example, Gibbs and Orton (1994) state that as groups can be formed on the basis of very different criteria (friendship, mixed-ability and so on), care must be taken when forming groups "as there are so many different intentions" (p. 109).

Because the research on written discourse in the mathematics classroom is relatively abundant, it will take too much space to describe them in detail. So, I will summarise, very briefly, the major findings and point to references that will give further details. Most of the relevant research has examined student-generated questions as well as journal writing, written explanations (with and without diagrams) and other forms of expressive writing (for example, Menon, 1996; Mett,

1987; Mildren, 1992; Morrow & Schifter, 1988; Rose, 1990; Selfe, Petersen & Nahrgang, 1986; Silverman, Winograd & Strohauser, 1992; Walter, 1988; Waywood, 1991; Wilde, 1991). The researchers are unanimous in their support of writing to learn mathematics, and list benefits such as the following: more student ownership of learning, monitoring and diagnosing of learning, lessening of student anxiety, enhancement of motivation and reflection.

Implications for Teaching

While it has not been possible to discuss comprehensively the factors linking mathematics and language in this chapter, nevertheless the following implications for teaching mathematics can still be made.

1. Discussing and teaching mathematics terminology enhances conceptual understanding of mathematics (Garbe, 1989; Miller, 1993).
2. Student-constructed problems help students understand mathematics better (Menon, 1995a, 1996; Silverman, Winograd & Strohauser, 1992; Walter, 1988).
3. Language learning ideas can be used to learn mathematics (Borasi & Agor, 1990; Capps & Pickreign, 1993; Greenes, Schulman & Spungin, 1992; Menon, 1995c).
4. Group discussions can be effective for the learning of mathematics if a number of factors affecting group dynamics are taken into account (Davidson, 1990; Gibbs & Orton, 1994; Wood & Yackel, 1990).
5. A variety of written tasks can be used in WTLM to cater for specific pedagogical needs in the mathematics classroom (Mildren, 1992; Rose, 1990; Selfe, Petersen & Nahrgang, 1986; Waywood, 1991; Wilde, 1991).
6. The language used in mathematics tests and word problems have a profound effect on the understanding and achievement of students (Clements & Ellerton, 1995; Hughes, 1986; Marinas & Clements, 1990; Newman, 1977).

Further Research

Let me first touch briefly on the methodology usually employed in the research on language and mathematics, before suggesting lines of inquiry for further research. Most of the recent research on language and mathematics has tended to be of a qualitative type. Perhaps this is to be expected, as the researchers are more aware that methods employed for the natural sciences—such as holding certain factors constant while varying others—may not be appropriate for dealing with human behaviour that may only be described by ambiguous and overlapping meanings (Chambers, 1991).

For example, although Selfe, Petersen and Nahrgang (1986) used an experimental design and multiple measures (including qualitative ones) for their study of college students' journal writing in the analytic and calculus courses, they report that the qualitative data gave them more indication of student understanding than did the quantitative data.

Yet another reason for using a qualitative paradigm when dealing with research on mathematics and language is because discourse is basically sociolinguistic, and sociolinguistic research is predominantly, if not exclusively, qualitative in nature.

While there has been a lot of research in mathematics and language, there is still much to be done. I now briefly discuss some general approaches that might be used for research on language and mathematics and also list some questions that might be suitable for further study.

Ellerton and Clements (1991) posited a theoretical framework to show the interface between mathematics and language "consistent with the multifaceted nature of relationships between mathematics, school mathematics, and language" (p. 19) encompassing culture, communication, curriculum theory, mathematics, classroom discourse, sociolinguistics, natural language, psycholinguistics, etc., and also mentioned some other models which might be used as a basis for discussion on language and mathematics. While the model proposed by Ellerton and Clements emphasises the complexity of the issues involved, it does not help directly in the design and interpretation of research in language and mathematics. However, it does alert would-be researchers in this area not to address issues in isolation.

Another line of inquiry is that suggested by Hicks (1995), who uses discourse analysis as a basis of her recommendations for research. She suggests an interdisciplinary focus, with members of a team of researchers working in collaboration. While her suggestion of working collaboratively holds promise, her suggestion of allowing and encouraging differing discourses that "could be seen as diverse paths to the construction of academic knowledge, rather than as evidence that individual children either did or did not 'possess' disciplinary knowledge" (p. 84) might not be well received by mathematics teachers and mathematics education researchers. Indeed, some might consider research on such discourse as evidence of trivial mathematics in, say, WTLM (for example, Caughey & Stephens, 1987; Ormell, 1992; Pengelly, 1990).

For example, if students were to be asked to write whatever they knew about fractions, some would use words and draw diagrams to accompany symbols for fractions, others might just give a personal narrative, perhaps identifying certain cartoon characters with specific fractions and so on. While not denying that students are learning something in the latter case, the disciplinary (mathematical) knowledge engendered by the personal narrative discourse genre might be called into question.

A more fruitful suggestion is to study "how discourse mediates the construction of knowledge in classrooms" (Hicks, 1995, p. 87). For example, questions on how to enhance mathematics learning through classroom discourse, what is the teacher's role, and what is the student's role in such discursive practices, are all worthy of study.

What is suggested, therefore, is that it might be useful to use the three lenses of teaching, learning and discourse-construction if one wants to answer questions on knowledge-construction in a specific discipline such as mathematics, and especially to studies on mathematics and language.

Some other questions worth pursuing might be:

1. How does justification of a process contribute to a deeper understanding of the concepts involved? And what sort of justification is legitimate in mathematics?
2. How does one progress from everyday English to a mathematical register?
3. What types of discourse changes take place during the construction of consensually-validated knowledge? What are the contributions of individuals to the discourse, and how do these change over time?
4. If mental imagery is considered a form of communication, how can it help develop mathematical concepts?
5. What communicative strategies are used by successful problem solvers?
6. What effect do computers have on discourse patterns for the learning of mathematics?
7. Which patterns of interaction (computer-pupil and pupil-pupil) facilitate mathematics learning?

Conclusion

In this chapter, I have drawn both from theoretical considerations and research studies, to explore the interrelationships between mathematics and language, as well as to suggest some implications for the classroom and for further research. While the topic of mathematics and language is too vast and complex to allow for a comprehensive treatment, especially when limited to a chapter in a book, I hope the interested reader (that elusive character!) has been given a glimpse of the many exciting possibilities in this field.

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The Role of Scaffolding in the Teaching and Learning of Mathematics

Jennie Bickmore-Brand

This paper will outline the components of scaffolding as they might be demonstrated in mathematics teaching. The discussion will be based on recent research that looked at two teachers teaching mathematics to upper primary students and the differences in their scaffolding techniques. The paper will analyse lesson transcripts, and share personal reflections by each teacher on the difficulties they experienced as they endeavoured to develop students' mathematical concepts and associated mathematical language.

Relevant Literature When Discussing Scaffolding

Scaffolding has been identified as one of the seven teaching/learning principles (Bickmore-Brand, 1989). The premise behind this teaching/learning principle is that in order for learning to take place, the learner has to connect the information in some way to what s/he already knows (Cobb & Steffe, 1983; Steffe & Cobb, 1988). The consequence of such a procedure is that knowledge will be idiosyncratically processed and stored by each individual (Bickmore-Brand, 1989). This principle has been well established by educational research (see for example, Bickmore-Brand, 1989; Bruner, 1983; Goodman, 1983; Kelly, 1955; Piaget & Inhelder, 1969; Vygotsky, 1962).

Connecting to What is Known

According to Kelly (1955), we construct our concept of the world and test it against the real world. This learning is a tensely active process, and new knowledge may often challenge our existing knowledge. Our ability to accommodate new information or experiences into existing conceptual structures will depend upon how dearly we hold onto our constructs (Bawden, 1985; Papert, 1980). Individual differences in how we view the world will also be influenced by how varied our life's experiences are and whether they provide an appropriate source for new constructs. Copeland (1984) recognised that children do not necessarily "see" or "remember" or "copy" what they are exposed to but reconstruct it for themselves. For this reason "truth" is considered to be derived from a variety of paths of action, and so any one construction may be equally as valid as another (von Glasersfeld, 1983).

Constructivists believe that the learner will feel a sense of "ownership" of the mathematical knowledge when it is actively linked to his or her own world—

"When someone actively links aspects of his or her physical and social environments with certain numerical, spatial and logical concepts a feeling of 'ownership' is often generated" (Ellerton & Clements, 1991, p. 56). With regard to early childhood teaching, Van den Brink (1988) recommended that "people" contexts rather than "object" contexts need to be employed. In this way, the concept development can be linked to a familiar context such as toys, animals or people.

Ausubel (1968), in the epigraph of his book *Education psychology: A cognitive view*, said "If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows" (p. vi).

Inability to Connect With the Learner

Clements and Del Campo (1990) are concerned that teachers should attempt to assist students to create links between the language and symbols of the mathematics studied in school, and the real-world context. The literature on the teacher or learner's inability to connect what the learner already knows to the classroom is well documented (Balacheff, 1991; Bero, 1994; Cobb, 1985; Ellerton & Clements, 1987; Pengelly, 1990; Sierpinska, 1996; Steffe & Cobb, 1988; van Dormolen, 1993). This is particularly the case in the learning environments of second language, languages other than English, Aboriginal and disadvantaged learners (Bishop, 1992; Carraher, 1991; D'Ambrosio, 1991; Enemburu, 1989; Saxe, 1988).

Behind the idiosyncratic processing of information lies the self-generation of rules. Through hypothesising and testing, the learner discovers or self-generates a concept (Green, 1988). Some of these self-made rules can interfere with the development of other rules which may have more currency. In the process the learner identifies which attributes can be generalised to newly encountered examples, and is able to discriminate between examples and non-examples (Tennyson & Park, 1980).

It is inevitable that the learner will make mistakes in this process but Goodman (1983) preferred to conceive of these errors as "miscues" or as "misperceiving" and considered them as opportunities, as a window might be, to look into the learner's mind. A similar idea was echoed in the constructivist literature pertaining to science teaching which was concerned with reconstructing students' misconceptions, or distorted preconceptions (Posner, Strike, Hewson & Gertzog, 1982).

Research on students' conceptions (Ausubel, 1968; Bawden, 1985; Kelly, 1955; Solomon, 1987) indicated that students' beliefs about the nature of the world is based largely on their experience and that even when presented with scientifically-based and rational explanations, many have difficulty "letting go" of their old beliefs and adopting a new perception. This notion of "cognitive conflict" (Ellerton & Clements, 1991) had its origins in Piagetian understandings of cognitive development. In order for students to change their ideas about a concept, they must either feel dissatisfied with their own knowledge, or become attracted to the benefits of entertaining a shift in thinking. Bickmore-Brand (1993) cites an

example where at the end of a lesson on volume using sand and water trays and containers of various sizes, a six year old student made the statement that volume was the knob on the radio you used to make the noise louder. This aptly illustrates the nature of schemas. Skemp (1986) described schemas in the following way:

A schema is of such value to an individual that the resistance to changing it can be great, and circumstances or individuals imposing pressure to change may be experienced as threats—and responded to accordingly. Even if it is less than a threat, reconstruction can be difficult, whereas assimilation of a new experience to an existing schema gives a feeling of mastery and is usually enjoyed. (p. 42)

Skemp was therefore suggesting that assimilation of a new concept is more likely to occur when the learner modifies an existing schema rather than reconstructs a concept completely.

Connecting With the Language of the Classroom

The linguistic term “register” has been described by Halliday (1978) as a “set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings” (p. 195). The language of the classroom contains a variety of registers.

Clements (1984) stated that children are likely to experience more difficulties with the language in the mathematics classroom than almost any other place they are likely to frequent. Discussions in the past decade (see Hunting, 1988; Marks & Mousley, 1990; Reeves, 1986; Watson, 1993) about the relationship of language to mathematics frequently deal with the highly specialised vocabulary of mathematics which involve a reinterpretation of everyday language (Bickmore-Brand, 1993).

Clements and Lean (1988) have emphasised the importance of learners making connections between familiar concepts, the formal mathematical language, and the manipulation of symbols (p. 222).

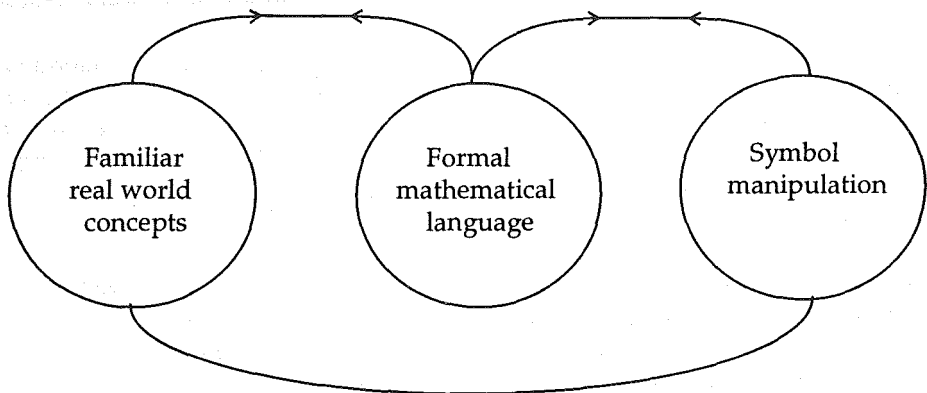


Figure 1: Establishing Links in Cognitive Structure (Clements & Lean, 1988, p. 222).

It is possible that unless these links are made students learn only fragmented pieces of information which are associated with what can be termed "school mathematics." Thus a major concern is that "if links are not initially drawn by children between their informal knowledge and the written symbols, children may develop separate systems of arithmetic, one that operates in school and one that operates in the real world, and they will not readily see the connections between them" (Grouws, 1992, p. 83; see also Carraher, Carraher & Schliemann, 1987; Cobb, 1988; Ginsburg, 1982; Lave, 1988). Research continues to explore how these relationships can best be developed within the school setting (for example, Bero, 1994; Bickmore-Brand & Gawne, 1990; Chapman, 1993; Clements & Del Campo, 1989, 1990; Gawne, 1990; Harris, 1991; Reeves, 1990; Watson, 1990).

Specialist vocabulary. Pimm (1987) described the notion of mathematics as a language as a metaphor for understanding mathematics in linguistic terms, for structuring "the concept of mathematics in terms of language" (p. xiv). His later work (1991) draws attention to the difference between written and spoken mathematics, the former relying on complex symbol systems and the latter which uses "natural" language, which assists the student to reflect on and to "conjure and control" the mental images in mathematics (p. 23).

Mathematics has its own specialised vocabulary as well as making use of standard words in non-standard ways. Students are often unable to adapt to the use of different semantic structures (Carpenter, Hiebert & Moser, 1981) or are distracted by an unfamiliar syntax (Goldin, 1992), even when similar information has already been provided.

Success in school mathematics. The above discussion is not to suggest that the language used in the classroom should be reduced to looking at vocabulary alone. Being successful in school is attributed by Lemke (1988) to learning how to think and talk appropriately and fluently, in the mathematics register.

It is not "superior intelligence" that makes for academic success in science or other fields. It is superior fluency in using the language of the subject: superior mastery of its genres, of its thematics, and of the techniques of combining these flexibly in practical use. (Lemke, 1988, p. 98)

As learners grapple with the terminology and associated concepts, they will continue to connect the new information with what they already know. Hodge and Kress (1988) noted that learners shift between what they describe as "less mathematical language" and "more mathematical" language. Chapman (1992) talked about being "less" or "more" mathematical as expressed as part of a continuum. Walkerdine (1982, 1988) refers to these shifts as being along the continuums of metaphoric and metonymic axes of mathematical discourse. The teacher in the role of scaffolder can assist students to make these shifts.

Scaffolding the Learner to Make the Connections

When one thinks of scaffolding the association of a framework, usually of interlinking steel pipes, or bamboo if you are in Asia, supporting the construction of a building or extension, may come to mind. This supporting structure is usually slightly above the actual building in progress and gets successively taller as the

construction develops. This can be a useful conceptualisation for the notion of scaffolding because it captures the tailoring of the support around the "needs" of the construction site.

Vygotsky (1962) is recognised as the founder of the label "scaffolding" when it is used to refer to the specific interactions which occur between adult/expert and child/learner. Scaffolding is based on the assumption that, with adult/expert support, learners can be stretched beyond what they might normally achieve without such support. He believed that the children would not have been able to develop in the same way just by virtue of age or maturation. Scaffolding is essentially a hand-holding strategy, tailored to meet the needs of the individual, and can be provided at any stage when assistance might be beneficial for the learner's development. There is no expectation that the learners will become independent learners independently.

Vygotsky used the term "zone of proximal development" to describe the way in which instruction can lead the child to focus on particular aspects of learning, in a joint problem-solving context which eventually can be independently handled by the child. The same idea can also be identified in the self concept work of the psychologist, Luft (1969), where he describes a dual process—of self-disclosure by an individual on the one hand and, on the other, of receiving feedback which helps individuals to reach their potential.

Since the aim of scaffolding is to build on what the child appears to know in order to stretch the child, the nature and form of scaffolding will vary, as will the time needed for scaffolding to be in place. The roles and responsibilities of the teacher and the student will vary, as will the joint construction of meaning, and the power of the teacher's modelling. The self-destructive nature of scaffolding enables the teacher or "expert" to continue to "raise the ante" of the discussion and regulate its predictability.

The Social Construction of the Teaching Situation

Some students are able to complete particular tasks when they follow along with the teacher, but they have difficulty tackling the same tasks for homework. According to Palincsar and Brown (1989), this difference could be attributed to the fact that many teachers provide prompts and clues as they complete tasks in the classroom. Although such prompts and clues may not be specifically directed to a particular student, but rather to the class in general, they lie within what Palincsar and Brown described as "band width." The students' learning is being socially supported by the culture established in the classroom by the teacher.

Chapman (1992) argued that "the relatively less successful student relies on the transformational language shifts between less mathematical language and more mathematical language" (p. 5). She provided examples of how the teacher might develop this with a student:

The essence of Arthur's answer is correct, but the teacher puts it into a more "proper" sentence structure. Stuart's term "the same" is restated by the teacher as "the same amount". The speakers are apparently making sense with each other as they develop a more mathematical way of talking. (p. 46)

In this example a dialectic model is operating where the student's learning is socially constructed. Austin and Howson (1979) made special note of the "language of the teacher," and referred to the importance of the teacher's role in helping to develop fluency in the mathematical language register.

Independent studies by Cairney (1987) and Zubrich (1987) examined how the adults/experts tailored their dialogue in an effort to create a shared construction of meaning. The above writers were influenced by Bruner's (1978) concept of scaffolding as a temporary framework providing a platform for the next step toward more "adult" communication. Bruner observed mothers who tried to prevent their children from "slipping back," by at the same time demanding more complex performances in their language (Lehr, 1985).

Ninio and Bruner (1978) have been particularly influential in their advocacy of scaffolding. They describe details of a dyad involving a mother with a young infant, in which the two are seen jointly constructing meaning, in spite of the major differences in language abilities of the two. In other words, the mother's scaffolding is at a level the child can manage and in the context of a presumably mutually satisfying interaction. As Holzman (1972) described interactions of this type: "The child finds out by the response of the adults what he is assumed to mean by what he is saying" (p. 321). Wells (1981) termed this a "negotiation of conversational meaning" (cited in Lehr, 1985, p. 667), when he observed the same interaction patterns in classrooms.

Scaffolding Can Act as a Framework or Platform for the Next Step in the Learning

Evidence that scaffolding is taking place is not always reflected in dialogue, but can take the form of a protocol. In fact, this is probably the most familiar form of scaffolding used in mathematics classrooms, for example, Polya's (1981) four-phase framework for problem-solving. Such a framework is used by some teachers as a scaffold to help students develop appropriate problem-solving skills.

A teacher's questioning of students tends to be one of the most readily available forms of scaffolding in the classroom. Apart from questions which are designed to test students' understanding, questions can, as suggested by Ainley (1988), perform three functions: (a) "structuring," which can activate students' existing knowledge in order to connect with new information; (b) "opening-up," which suggests further exploration such as "What would happen if ...?"; and (c) "checking," which encourages students to reflect on their own logic and processing such as "Do you agree with ...?" (p. 93). Ultimately the learners may be in a position to ask these questions of themselves or other learners, which leads into the next discussion on "the switching of roles of learners and teachers."

Interactionists (see, for example, Bauersfeld, 1988, 1992, 1995; Voigt, 1985, 1994, 1995) discussed the various assumptions, patterns and routines occurring during classroom discourse. Certain kinds of classroom interactions, such as "funnelling," focusing, reciting and "concrete-to-abstract" practices can have a profound effect on the quality and extent of the learning occurring in the classroom (Bauersfeld, 1995; Brousseau, Davis & Werner, 1986; Bruner, 1996; Krummheuer, 1995; Voigt, 1985, 1995). When these classroom interactions are sensitive to the learner's

conceptual understandings and build on these, they have the potential to act as a scaffold for the learner.

The Switching of Roles of Learners and Teachers

The notion of "turn taking" between teachers and learners has been explored by Stern (1975) and Snow (1976). Wells (1981) also noted this feature of turn taking and the instructional role the adult takes on during a dialogue. It is hoped that the questioning and responses of the "expert" will enable the learner to eventually internalise this form of dialogue and ultimately ask these questions of themselves (Scollon, 1976; Staton, 1984). In this way learners have been able to draw upon the teacher's language as a resource (Kreft, 1983-84) and the scaffold can self-destruct.

Tizzard and Hughes (1984) pointed out that the role of the adult/expert/teacher should be viewed as a flexible one in which the learners may be encouraged to adopt the teacher's language while the adult takes on a more passive and non-directive role.

Taylor's (1992) study of a Year 12 mathematics teacher noted the difficulty a teacher had in adopting a "teacher as learner" role, and attempts to refine his "teacher as informer" role. According to Taylor, this teacher was unable to adjust to the role of "teacher as learner" because of personally constraining beliefs about his "technical curriculum rationality" which appeared to keep him in the role of "teacher as controller." There was, however, some shift from "teacher as transformer" to "teacher as interactive transformer." Alro and Skovsmose (1996) emphasised that the negotiation of meaning in a classroom where the teacher has an "absolutist" view of mathematics may be more concerned with students clarifying the meaning they suppose is in the mind of the teacher or textbook, rather than constructing mathematical meaning for themselves.

In classrooms where co-operative teaching/learning procedures have been adopted, students' dialogue attempts to bridge the gap between their peers' knowledge and that of the content being presented (Dansereau, 1987; Singer, 1978).

The Teacher Regulates the Level of Difficulty for the Learner

Research carried out by Snow (1983) and Thomas (1985) analysing adult-child discourse suggested that an adult will often continue and extend the topic that the child introduces. In order to regulate the level of difficulty of the interaction for the child, the adult will use a predictable structure to the dialogue to allow for more complex concepts to be discussed (Clark, 1976; Snow, 1983; Thomas, 1985). The adult will be aware of what the child can do and will insist on certain levels of performance, they "raise the ante" and gently increase the challenge. Other writers have contributed along the same lines in regard to this idea of adjusting the level of difficulty between teacher and students during classroom interactions (see for example, Bruner, 1983; Cambourne, 1988).

The Self Destructive Nature of Scaffolding

Bruner (1986) described the teacher/student interactions of scaffolding as "the loan of consciousness that gets the children through the zone of proximal

development" (p. 132). Inevitably the adult/expert gradually withdraws, and as the child/learner develops genuine understanding of the concept, the scaffolding is taken away or no longer used—it self-destructs (Cazden, 1983).

There are differing views on who is in control of removing the scaffolds. Cambourne and Turbill (1986) perceived that the control for the learning is exercised by the learner and not the adult (this view was influenced by Graves (1983) and by Harste, Woodward and Burke (1984)): "As learning occurs the scaffolds are removed by the children and others serving a different function may be erected" (Cambourne & Turbill, 1986). Alro and Skovsmose (1996) described classroom discourse as being very much in the control of the teacher, who determines what children may discuss in the mathematics classroom and frames the knowledge and how it will be handled.

Regardless of whether it is the child who is in control of the self-destruct aspect of scaffolding or the adult, there is no doubt that some modelling of the language will have to be present in the child's environment. As Herber and Nelson-Herber (1987) commented: "Students should not be expected to become independent learners independently. Rather they should be shown how to become independent, and this showing how should be a natural part of instruction" (p. 584). Noddings (1990) suggested that although students inevitably perform constructions, the mathematics that they produce may not necessarily be adequate, accurate or powerful.

Predictable Routines as Part of the Classroom Culture

In an attempt to maximise the effectiveness of scaffolding, Applebee and Langer (1983) proposed that the structure of an activity be made more explicit. This can be achieved if the context is predictable (Bruner & Ratner, 1978; Cambourne, 1988; Ninio & Bruner, 1978; Wells, 1981) and the questioning and modelling is structured so that the children have an opportunity to internalise it, and can eventually function without the external support. The professional development package for secondary teachers titled *Stepping Out* (Education Department of Western Australian, 1996) provided detailed guidelines for teachers to develop scaffolds in this way. For example, these guidelines provided writing frameworks to help teachers make explicit what they expect of students in relation to a given piece of work. The guidelines also suggested ways in which teachers can provide support for students as they attempt to interpret text.

Mousley and Marks (1991) discussed the use of similar writing frameworks in the mathematics classroom. For example the "procedure" genre (Martin, 1985) is recognisable in mathematics texts—"First write ...," "Now take away ...," "Have a look to see which pronumeral is easiest to ... " (p. 5). Martin (1985) commends the practice of having students write in these genres because "such a task enables students to clarify both the nature of mathematical processes and the logical orders in which these might be carried out ... Students need to develop greater control of the more sophisticated expository genres that are valued in mathematical culture" (p. 6).

Halliday (1981) has included "structure" as one of his five criteria for the application of scaffolding in the classroom, where he believes that the framework

of an activity should be predictable enough to provide support for the students. In a similar way, Halliday advocates that the modelling and questioning components of an activity should take on a recognisable "routine."

The notion of "routines" was discussed by Trevarthen (1980) when he pointed out that "the routines of action and the rules behind them are accepted because of a co-operative motive, but they do not create the motive" (cited in Searle, 1984, p. 482). Routines should not be used to justify making children restructure their experience to fit their teacher's structure. Learners should have transformational freedom in deciding how to arrive at and express the concept (Chapman, 1992).

The power of the scaffold is, first, in the perception of the teacher's role and relationship with the learner. Second, but no less important, it is in the teacher's own conceptual understanding of what is being taught and his/her ability to assess the students' response in relation to their concept development. A close look at two different scaffolding styles in this chapter aims to explore how this might look in two different classrooms.

An Investigation of Scaffolding Practices in Two Classrooms

The Learner-Centred Classroom

The first teacher in the study, Lyn¹, could be described as learner-centred. Her teaching approach towards mathematics was an integrative thematic programme and dealt with topics about which her students were keenly interested, for example, they set up their own small business making art and craft products for sale during lunch times, they planned the most cost efficient route and form of transport for their class trip to a campsite at Collie some 250 km away in the country, and they ran the school Sports Day as a Mini Olympics and drew to scale the marking layouts for the events to be held on the school oval. The topics could last up to a whole term (13 weeks) in duration and crossed into other subject areas during the day. Lyn believed that connecting the curriculum to the background of the learners was important. She wanted her students to see that school, and in particular mathematics, was relevant to their own everyday lives. She wanted to ensure that the links that Clements and Lean (1988) discussed were explicit in her classroom.

The following transcript has been taken from a small-group discussion in which Lyn tried to develop students' understanding of fractions so that they could solve a problem which a small group was having with working out how many sandwiches would be needed for a class party. Each time a child had not quite grasped the idea she tried to give a "concrete" demonstration of the idea.

Teacher-Lyn stops at one group who are working on sandwiches.

Teacher-Lyn: So you'll have to write something about bread. How many loaves of bread you will need.

1. In order to more readily identify the difference between the two classrooms each has been assigned a different font, i.e., Lyn and Michelle.

Joe: We're going to have sandwiches and cut them like this (makes 2 diagonal cuts with his hands). How much bread will we need?

Teacher-Lyn: Right. How many slices in a loaf of bread?

Ben C: 24.

Teacher-Lyn: I was going to say 24 as well. Right so if you get 24 and you put one on top of the other (demonstrates with her hands) so that's ...?

Joe: Twelve.

Teacher-Lyn: Twelve Sandwiches. If you do them into ...? (makes a diagonal cut with her hands).

Joe: Halves.

Ben C: No quarters (makes 2 diagonal cuts)

Teacher-Lyn: Quarters. So there'll be 4 lots of 12...

William: 48.

Teacher-Lyn: 48 sandwiches.

Ben C: Not everybody would eat it so you'd be able to have 24 quarters.

Teacher-Lyn: Is that for 2 loaves of bread?

Joe: Mrs?

Teacher-Lyn: Remember we said there were 12 slices (demonstrates by drawing square with her fingers), and we cut those 12 into 4

William: 24.

Teacher-Lyn: You've got 24 slices but when you make them into double sides (demonstrates with hands) that reduces it to how many stacks (demonstrates with hands) of bread?

William: 12.

Teacher-Lyn: Right. Now if you cut that 12 into lots of 4 how many is that?

William: 48.

Teacher-Lyn: And you've got 2 loaves of bread. (pause) What's 48 plus ...?

Joe: 48.

Ben C: 90, 92

Teacher-Lyn: Not 92, 48 and 48. What's 8 and 8?

William: 16.

Teacher-Lyn: 16. So it's ...?

William: 96.

Teacher-Lyn: And if there's 32 of us approximately, how many will each child have? Approximately?

William: They'll have about half.

Teacher-Lyn: How many 30s in 96?

Ben C: 3.

Teacher-Lyn: About 3. Would it be 3 whole sandwiches or just three quarters of a sandwich? So each can have about one round of sandwiches.

Later, the group reported back to the class:

Teacher-Lyn: So you decided what to do about your sandwiches?

Ben C: Some people don't like sandwiches, but we decided that you could get $\frac{3}{4}$ of a sandwich each and we needed 2 loaves of bread.

Lyn has attempted to develop the students' fraction knowledge in a context which is purposeful for the students. In addition, the concept has been tempered with a critical numeracy understanding of the skills they are using. In the end the final mathematical calculations are tempered by the realities of life—"Some people

don't like sandwiches."

Lyn operated in a "teacher as interactive transformer" (Taylor, 1992) when she appeared to take her cue from the students. When the students signalled some need—"How much bread will we need?" (Joe), she seemed to clarify the need in the first instance—"How many slices in a loaf of bread?" (Teacher-Lyn), and then suggest a further application which might extend the students' thinking—"So if you get 24 and you put one on top of the other?" (Teacher-Lyn).

The timing of Lyn's assistance was usually after she had allowed the students to develop their own approaches and arrive at a finished product at their own level. The students had been working in small groups on a food item of their own choice in planning to cater for their class party.

Lyn's dialogue leads the students to focus on a particular aspect of learning (addition of fractions) and uses a joint-problem-solving context which eventually enables the students to handle the task independently (Vygotsky, 1962). Lyn's questioning covers the three broad functions suggested by Ainley (1988): "structuring" when she initially draws upon what the students know, "opening-up" in order to assist the students to consider solutions, and "checking" in order to reflect on the logic of what they have been thinking. This dialectic model where the teacher's language is responsive to the students' is further developed below, drawing on the scaffolding features (see italics) described in Bickmore-Brand and Gawned (1990).

Lyn "structures" the dialogue by *restating the task* when she opens with "So you'll have to write something about bread. How many loaves you will need." Notice the student's (Joe) response is assertive and he is obviously clear about the task at hand. The teacher is there to provide relevant information and does not play a "teacher as controller" (Taylor, 1992) game of withholding information that the students feel they have to worm out of her. Lyn provides "structure" by *elaborating on the context* and accompanying her discussion with gestures (makes a diagonal cut with her hands) when ambiguity might have occurred in the communication. She uses *clarification* in order to ascertain whether she is on the same wavelength as the students or not, "Quarters. So there'll be 4 lots of 12." She "opens up" by including a *rhetorical question* as a prompt, "Is that for 2 loaves of bread?" Lyn "checks" by *reflecting back* to the students their thinking processes to that point in order for them to stand back a little from the ideas and analyse them—"Remember we said there were 12 slices ... " Lyn is jointly constructing the solution with the students. Where the students do not seem to be moving forward so readily she *requests information*—"And you've got 2 loaves of bread (pause) What's 48 plus ... ?", which then enables the students to continue to process the ideas. She provides *information for comparison* so that students can decide on the logic of the choice—"Would it be 3 whole sandwiches or just three quarters of a sandwich?" Note, however, there is an absence of mathematical terminology except for the use of "diagonal." The language Lyn uses emphasises everyday associations with which the students would probably be familiar, e.g., "one round of sandwiches."

Trevarthen (1980) pointed out that routines can act as a scaffolding framework for students. Lyn would often provide a framework in the form of a routine for the students to build their confidence. In the transcript example the students were planning for a class party which involved the repetition of adding up lists of

grocery items under certain categories, e.g., chips, drinks, sweets, etc., and therefore had an in-built routine. Trevarthen (1980) was concerned that routines were acceptable in classrooms because of their cooperative motive not because the routines created the motive.

This paper began with the discussion of the individuality of the construction of knowledge. Lyn valued the idiosyncratic way students solved problems and discussed how frameworks too can be flexible.

It's very important to provide them with a framework, but in maths it's difficult because you can provide them with the framework because there is not really just one right way in coming to that, there may be for instance in addition, there may be different ways a child sees to calculate, different ways they remember their number facts, different links that they make, so it's important that you model and scaffold a variety of ways.

Lyn encouraged her students to work in pairs and to use their own methods. In this way students who had recently learned a skill were in a position to mentor other students and as the following comment shows, it wasn't always the same students being the "expert."

Just recently a very slow child, a child who has difficulty in learning could actually make a rectangular and triangular prism, and the very bright ones in maths, in number, could not get it to work, and so the less able student normally who was seen by the group as the less able student became the peer tutor and so that was wonderful—and this child doesn't speak very much at all. But I could say "Go and see James."

Although whole class demonstrations of frameworks or protocols by her alone were rare it can be seen by the interaction patterns in these lessons how inclusive Lyn was of the class members (see Table 1). The pattern of interactions also indicates the hand-holding or joint-construction nature of scaffolding.

Lyn's focus was on assisting her students to access the content of her classroom and therefore her choice of topics and her interactions with her students attempted to be inclusive.

The Content-Centred Classroom

The second teacher in the study, Michelle, could be described as content-centred. She believed that students needed to be prepared for high school and University study. Michelle's lessons progressively covered the content set out by the WA Education Department for the upper primary year level she was teaching. The children were ability grouped in an attempt to streamline the delivery of the content. Michelle had one of the top ability groups. The students covered topics within the major strands of Number, Space and Measurement, e.g., equivalent fractions, regular polygons, and conversion of centimetres to metres. These topics at times lasted one lesson period (45 minutes) or at others were built up over several days.

Table 1

Student-Teacher Interactions in Whole Class Discussions: Lyn
(taken from field notes class maps from 7 lessons)

Students	4 Mar	11 Mar	5 May	13 May	10 Jun	30 Jul	5 Aug
Ben B	✓✓	✓	✓	✓	✓✓✓		
Will B	✓	✓			✓	✓✓	✓✓
Ben C		✓		✓	✓✓	✓	✓
Olivia D	✓		✓✓	✓	✓✓✓		✓✓
Ben E		✓		✓	✓		✓
Louise G			✓		✓	✓	✓
Chris G		✓	✓		✓		
Reuven G	✓✓	✓✓	✓	✓✓		✓✓✓	✓
Stephen H	✓✓✓	✓✓	✓✓✓	✓✓	✓✓	✓✓✓	✓✓✓
Dean J	✓✓	✓	✓	✓		✓	
Matthew J	✓✓	✓	✓	✓		✓	
Melanie K	✓✓✓	✓✓	✓✓	✓✓	✓✓✓	✓	✓✓
Aaron K					✓	✓✓	✓
Melissa L	✓		✓✓	✓✓		✓	✓
Stuart M	✓	✓✓✓	✓	✓	✓✓	✓	✓
Carmen O	✓		✓	✓	✓	✓✓	✓
Oliver P	✓✓		✓✓			✓	✓
Victoria P	✓✓✓	✓✓✓	✓✓	✓✓✓	✓	✓	✓✓
Blake S	✓✓✓		✓	✓	✓	✓✓	✓✓
Brenda S	✓✓	✓✓	✓✓	✓✓	✓	✓✓✓	✓
Mark S			✓		✓	✓	✓
Paul S	✓✓✓		✓✓✓	✓✓✓	✓	✓✓✓✓	✓
Paul T	✓✓✓		✓	✓		✓	✓
Joe W	✓	✓			✓		✓
Tim W	✓✓✓	✓		✓	✓✓	✓	
Chris W			✓	✓	✓	✓	
Number of students interacting	19	15	20	19	20	21	20
Total interactions	39	23	30	28	30	34	27

Michelle specifically referred to mathematical processes in her suggestions to assist the students. She believed that being successful in school could be enhanced by thinking and talking appropriately in the language of the mathematics register (Lemke, 1988). The transcript from one of Michelle's lessons begins with dialogue which related to a difficulty the class had been having with equivalent fractions. Her remediation hints focused on trying to have the children grasp the pattern or protocol she had in mind in her attempt to reshape the (mis)conceptions students may have had. This is particularly evident in Michelle's responses to Georgia.

This excerpt from a lesson transcript shows how Georgia continues to retain her misconception of the concept even when Michelle is attempting to lead her

through in what Michelle feels is a systematic and logical way. The lines which show Georgia's struggle have been coded (###). Note how Michelle responds to Georgia at these points.

Teacher-Michelle: All right, are you ready? Right girls eyes this way please. What we will be doing today is a continuation of what we did Monday and yesterday. Yesterday we were looking at equivalent fractions and we were working out how to find equivalent fractions of a simple fraction such as a third without actually having to draw those large graphs every time you need to work out an equivalent fraction. What is that way that we came up with yesterday? Do you remember we saw a pattern between the different numbers? For instance we could tell instantly how many sixths is a third. Gabrielle?

Gabrielle: Two.

Teacher-Michelle: And how did you get two sixths?

Gabrielle: Um two thirds is six and once two is two?

Teacher-Michelle: Right you realised that whatever the denominator was multiplied by, the numerator was then multiplied by the same number. One times two is two. What else did we say to prove that one third does actually equal two sixths? We came up with a way to prove that two sixths was equal to a third. Do you remember what we said yesterday? We said that we know if I had two sixths of a cake I would have just the same amount as someone that had one third... and what was that way Georgia?

Georgia: Because in the top number goes into the bottom number three times and in the second fraction the top number goes into the bottom number three times as well.

Teacher-Michelle: That's in this instance but it doesn't always happen. I was actually talking about this little thing here that we talked about yesterday, what did we say about that number Angela?

Angela: That you times it by ... you can ... it's a whole number and you can say two ... it's a whole number ...

Teacher-Michelle: Right you're saying two halves of two is a whole number right so you're saying two halves is one. One third multiplied by two halves equals two sixths. And all we are doing is multiplying a third by?

Class: One.

Teacher-Michelle: And we know whatever number we multiply by one we end up with the same number. Because ten times one is ten. Twenty- five times one is? ... Pardon?

Class: Twenty-five.

Teacher-Michelle: OK So that's what we looked at yesterday. Now I know some of you had a little trouble with that so the plan today is to put you to work at your desk and then I'm going to mark the homework that you did last night and Mrs Bickmore-Brand will also mark your homework last night, individually so that we can see that you really understood what we talked about yesterday.

Child: But I didn't get it.

Teacher-Michelle: That's right you can talk about that when you talk about your homework. Right for today, we are going to do the same thing in reverse OK? We talked about one third and two sixths being equivalent fractions. Can anybody tell me

other fractions that are equivalent to one third? ... Right? Rebecca?

Whole class on equivalent fractions continues for a further 20 minutes.

Teacher-Michelle: If any one would like to see me having marked those eight come over to me now. Georgia, come and see me. Bring your work. Everyone else quite happy to continue along? ... Come and sit down let's have a look Georgia. What is the major problem?

###Georgia: Oh I haven't got one. I wanted to ask you something else. Shall I do it?

Teacher-Michelle: I'll just mark your homework for you. How did you find this.

Georgia: Oh it's easy. I think.

Teacher-Michelle: Did it take you very long.

Georgia: No. About five minutes.

Teacher-Michelle: Well tell me your system?

Georgia: Well. I think I do it differently, but if you say twenty four, Three goes into twelve four times, so times four by four and you get sixteen.

Teacher-Michelle: Right so whatever you have done to the denominator you ...

Georgia: Do to the numerator.

Teacher-Michelle: Have done to the numerator.

###Georgia: Oh gosh.

Michelle: Oh dear what have we done down here? Let's have a look. One half is equivalent to two quarters which is equivalent to three sixths.

###Georgia: And they're all halves and I thought. Oh no.

Teacher-Michelle: Tell me how you came about six eighths and that will identify the problem.

Georgia: I put two, four, six, eight, ten, twelve. And seeing's how you times it by two there and timesed by two to get six and then added two and then added two. I did it completely wrong.

Teacher-Michelle: Why did you add two?

###Georgia: I don't know. Because I was supposed to be timesing two. No. Yeah. No.

Teacher-Michelle: Can you tell me what each of these are supposed to be equivalent to in the lowest term?

Georgia: A half.

Teacher-Michelle: A half, good girl, so that's the lowest term fraction. So six eighths we know is not equivalent to a half.

Georgia: It would be six twelfths, eight sixteenths and ten twentieths.

Teacher-Michelle: Eight sixteenths and ten twentieths OK. But those aren't the next three equivalent fractions, because the next one is going to be ... eighths. You've got halves, quarters and sixths, and so the next one would need to be eighths. The bottom line, the denominators are going up in multiples of?

###Georgia: Two that's how I did there. I thought.

Teacher-Michelle: So this one then should be tenths. And this one? Georgia: Twelfths.

Teacher-Michelle: Twelfths. Let's fill them in. How many eighths is equal to a half?

Georgia: Four.

Teacher-Michelle: How many tenths is equal to a half?

Georgia: Five.

Teacher-Michelle: And? Six twelfths. Can you understand that? So let's go to this one look at the denominator and the pattern that the denominator is making. What are the next three denominators?

###Georgia: It would be twenty.

Teacher-Michelle: No it's going five, ten, fifteen?

Georgia: Twenty,

Teacher-Michelle: Twenty, twenty-five?

Georgia: Thirty.

Teacher-Michelle: Mm. The lowest term fraction that we're looking for?

###Georgia: Two, four, six, eight, ten, twelve, fifteen?

Teacher-Michelle: Let's have a look at it. It's making a pattern, but is eight twentieths equal to two fifths?

Georgia: (pause)

Teacher-Michelle: Let's work it out. How many fives in twenty?

Georgia: Four

Teacher-Michelle: Four. Two fours are?

Georgia: Eight.

Teacher-Michelle: Is your pattern working? Let's check for the next one. How many fives in twenty five?

Georgia: Five.

Teacher-Michelle: Five tens?

Georgia: Fifteen. Goes. Is ten.

Teacher-Michelle: Good, so can you tell me the next answer?

Georgia: Um five into fifteen, thirty goes two, two times six is twelve.

Teacher-Michelle: Good your pattern worked.

###Georgia: It wasn't really what I was thinking.

Teacher-Michelle: It doesn't always work in a nice pattern like that. So you need to be careful and always relate these equivalent fractions back to the lowest term fraction. OK. Understand that? I'll leave the next four to do by yourself tonight and show me tomorrow.

Using Ainley's (1988) questioning framework together with Bickmore-Brand and Gawned's (1990) scaffolding features (see italics), Michelle's dialogue pattern can be revealed to be qualitatively different to Lyn's.

Michelle initiates the whole class discussion by "structuring" and *stating the task* at hand—"What we will be doing today is a continuation of what we did Monday ..." As the interaction between Michelle and the students unfolds she uses clarifying statements to reflect back to the students what they have been trying to express. Due to the importance to this task of building on previously discussed processes Michelle repeatedly "structures" during the interaction—"Do you remember what we said yesterday?" ... and later "Right for today, we are going to do the same thing in reverse OK?" and when working with Georgia—"Tell me how you came about six eighths and that will identify the problem." Because there is a protocol that is being developed with the students,

examples of "opening up" have limited parameters—"I would have just the same amount as someone that had one third ... and what was that way Georgia?" and later "Can anybody tell me other fractions that are equivalent to one third?"

There is a clear message that Michelle is trying to assist her students to approach the task in a certain way that will reflect a "system" and the focus on the task is more important than the student's (mis)conceptions—when Georgia says "Two that's how I did there. I thought," Michelle continues to develop the pattern she is working with. Georgia appears to momentarily appreciate Michelle's pattern but when asked "So let's go and look at the denominator and the pattern that the denominator is making. What are the next three denominators?" (Teacher-Michelle) Georgia once again flounders. Ausubel (1968), Bawden (1985), Kelly (1955) and Solomon (1987) all commented how difficult it is for people to let go of their previous beliefs and constructs.

Although this class was ability grouped, there were still students working at different levels. This dialogue took place after considerable classroom work on fractions throughout the year using a certain system, which clearly, now in August, after another series of lessons on fractions, students are still struggling with. In this transcript Michelle started at a point where many of the students were experiencing difficulties. She used this base of the students' understanding to develop scaffolding to support *their* concept development. As the lesson progressed, however, Michelle attempted to ensure that the students' understanding corresponded to *her* understanding.

Michelle was gently trying to impose her own approach to this fraction topic. Implicit in this tactic was the assumption that she could persuade Georgia to understand the way she was using pattern to develop equivalent fractions. Towards the end of the dialogue she was using expressions like "Good, your pattern worked" (in spite of Georgia's protests that "it wasn't really what I was thinking"). Thus although Michelle appeared to take account of the child's approach early in the dialogue, her later reference to "your pattern" did little more than reinforce the fact that she was imposing hers.

Wells (1981) would describe this interaction as a "negotiation" of conversational meaning. However, it reflects the interaction patterns Lehr (1985) refers to in classroom discourse and recognisable in Taylor's (1992) terms where the teacher operates with "technical curriculum rationality." Michelle herself admitted "I did feel very constrained to the syllabus to a very large extent. I think maths being very sequential and developmental, you do need to make sure prior knowledge is developed before you work on to more complex concepts. Therefore it's important to follow and complete the year's work."

In a classroom where the content takes precedence over the individual's processing the teacher-student interactions reveal certain patterns. Table 2 indicates that the number of students interacting in whole class discussions in Michelle's class is limited when compared with Table 1 of Lyn's class. The joint construction in Michelle's class is not concerned with building a shared understanding which takes into account students' particular interpretations. Michelle overtly altered the

children's everyday non-mathematical language to more precise mathematical terminology. She also introduced the more abstract terminology of "lowest term" and reinforced "denominator" and "common factor."

Table 2

Student-Teacher Interactions in Whole Class Discussions: Michelle
(taken from field notes class maps from 7 lessons)

Students	28 Jan	16 Mar	6 Apr	3 Jun	9 Jun	18 Jun	10 Jul
Carrie B		✓		✓✓✓	✓		
Elizabeth B	✓✓	✓		✓		✓✓✓	
Genevieve C	✓						
Rebecca C	✓✓	✓				✓✓✓	
Gabrielle C	✓✓	✓✓					
Emma C		✓✓		✓			
Angharad D	✓	✓✓		✓			
Susan G		✓✓✓✓	✓				
Gabrielle H				✓		✓✓	
Emily K			✓✓		✓✓		✓✓✓
Olivia K	✓✓			✓✓	✓		
Carla L	✓	✓✓			✓✓		✓
Amy M		✓					
Elizabeth M				✓✓	✓	✓	✓✓
Angela M		✓			✓	✓✓	✓
Georgia M	✓✓✓	✓✓✓	✓	✓✓✓✓	✓✓✓✓	✓	✓✓✓
Adeline M							
Naomi M		✓✓					
Linda N		✓	✓			✓✓	
Carrie P			✓		✓✓✓✓		
Catherine R	✓✓	✓✓		✓✓			
Rebecca S	✓✓	✓			✓		
Lynleigh S	✓		✓	✓✓	✓		
Kylene T		✓✓✓		✓✓		✓✓	
Chantelle V	✓	✓	✓	✓✓✓✓			
Melissa W	✓	✓✓	✓✓				
Rebecca W	✓✓						
Alana W				✓	✓✓	✓✓	
Leanne T	✓						
Number of students interacting	15	18	8	13	11	9	5
Total interactions	24	32	10	26	20	18	10

Specific frameworks for problem-solving were a feature of Michelle's classroom discourse. The children had in the back of their books a checklist for problem-solving.

1. What is the problem?—state the problem in your own words.
2. Solving the problem—all the working out.
3. Solution—sentence. (Polya, 1981)

In their text book there was also a similar model (Addison Wesley, p.T8).

1. Understand the question,
2. Find the needed data,
3. Plan what to do,
4. Find the answer,
5. Check back.

The students were regularly taken through lessons which developed strategies including the following.

1. Guess and Check.
2. Draw a Picture.
3. Make an Organised List.
4. Make a Table.
5. Work Backwards.
6. Look for a Pattern.
7. Use Logical Reasoning.

At the end of twelve months of classroom observations in both classes the students were given a novel question in a Post Interview to solve out aloud ("On what day and in what year will your 21st birthday fall?"). There was no discernable evidence that Michelle's class had been taught any specific strategies within these problem-solving frameworks. In fact the results suggested a highly idiosyncratic approach to problem solving for both classes.

The Role of Teacher as Scaffolder in Developing the Mathematical Register

The ways in which each teacher has chosen to scaffold her students are qualitatively different. Both teachers have relied on constructing situations in which they can provide a framework for mathematical skill and concept development. They both use the students' peers in the construction of the scaffolding, although this has qualitative differences. Both teachers started with a problem or concept which was targetted specifically for the child/class.

The differences lie in how each scaffolding dialogue was constructed. Michelle signalled her preferred pathway for the development of a concept, and although clearly aware of a child's misunderstanding continued to drive on with a predetermined method for approaching that skill or concept. Lyn, however, continued to shape and develop a skill or concept down a pathway which is jointly constructed between her and the child/class. She worked with their ideas as they came up, for example, Joe wanted the sandwiches cut diagonally, and Ben suggested quarters. Her explanation of a skill or concept continued to be reworked in an effort to refine the communication of the idea rather than presenting a system for approaching the task.

Michelle continued to "raise the ante" in terms of regulating the difficulty of the concept development. Increased demands were placed on the students to apply their conceptual understanding of fraction concepts. Lyn tended to focus on the immediate conceptual difficulty the children were facing as they tried to solve the problem they were tackling.

These examples are limited in what they can demonstrate in terms of the degree to which the children are developing their mathematical language and use of mathematical proformas. Michelle made greater use of mathematical terminology during her exchanges with students. She was observed not only introducing new vocabulary, and explicitly using this mathematical vocabulary in her own discourse, but she monitored the mathematical vocabulary students were using, as can be seen in the following interaction.

Michelle: Now of those fractions this one's special and it has a special name called? (Pause) Does anybody know?

Child: A lowest term?

Michelle: A lowest term fraction. Why do you think it would be called a lowest term fraction?

Child: Because it can't go any lower than that?

Michelle: The denominator can't go any lower than the three.

Rather than the teacher generating a joint construction of mathematical meaning with the students through her discourse, the mathematical language is presented by Michelle as an "expert" who controls the language being used by the students. This, in fact, had a positive result when the students were asked to perform the Progressive Achievement Test and Placement Test J assessments. Michelle's students seemed less hindered by the language being used in the test items than Lyn's students.

Lyn, however, tended to focus on mathematical language when it appeared to be responsible for the children's misunderstanding. Sometimes Lyn was able to identify an everyday word for which the children needed to adjust their range of meanings. The vocabulary the teacher introduced was readily accessible to the children, e.g., diagonal, halves. The students in Lyn's class did have difficulty with some of the language on Placement Test J. For example, the majority of her class had difficulty with items involving the solving of algebraic-type problems containing subtraction, multiplication and division ($[] - 10 = 9$; $10 \times 4 = [] \times 10$; $[] \div 2 = 9$), and with the question "What is the product of $\frac{1}{4}$ and $\frac{1}{3}$?" Carpenter, Hiebert and Moser (1981) also observed the inability of students to use unfamiliar syntax of mathematics even though the processes would have been familiar to them.

The lack of emphasis upon the language of mathematics was quite obvious in Lyn's classroom discourse. Although she was keen to clarify any confusion students may have had in everyday words that have different interpretations in different contexts (for example, timetable) Lyn was less overt in her use of mathematical vocabulary. During her interaction with the students, her words shifted between everyday and mathematical language, as this lesson transcript excerpt shows.

Lyn: And if there's 32 of us approximately, how many will each child have? Approximately?

William: They'll have about half.

Lyn: How many 30s in 96?

Ben C: 3.

Lyn: About 3. Would it be 3 whole sandwiches or just three quarters of a sandwich? So each can have about one round of sandwiches.

In the whole class discussion which followed the above interaction, Lyn could be seen giving the students the freedom to adopt her suggestions or those from her classmates, or to continue to negotiate with them (about the number of sandwiches required). In general, Lyn's classroom discourse showed her to be less directive and more passive, regularly allowing students to take the lead.

Overall, Michelle and Lyn adopted different scaffolding styles, with Michelle diagnosing a child's misunderstanding of a mathematical skill or concept and imposing strict mathematical language. She seemed unable or unwilling to adjust her own language or methodology to embrace that of the child's, or to allow alternative ways of approaching a solution. Lyn tried to use scaffolding to help to shape the skill or concept around the child's understanding. Lyn's own language, however, did not always lead students to a refining of their mathematical language.

This chapter has therefore identified a dilemma concerning each teacher's approach. If scaffolding takes the form of the language of the teacher taking on the learner's language and developing it, through interaction, so that the learner becomes increasingly more conversant with the mathematical register (Halliday, 1978), then neither teacher in this study appeared to be doing this. Michelle used the mathematical register, but this was imposed onto the learner's own language. Lyn was sensitive to the learner's own vocabulary but rarely helped students to develop this into a mathematical register. Clearly the mathematical language was not signalled to the students as being more valued than their own language. This was not the case in Michelle's class, where the mathematical register was valued as the preferred discourse.

Bickmore-Brand and Gawned (1990) identified three types of scaffolding techniques:

- Task Focused—this style tends to adhere to the formal requirements of the task.
- Child Focused—this style supports the children in whichever way they choose to explore the task.
- Multi-Focused—this style provides support in order to meet the needs of the particular child at each point during the task when a shared focus is seen to be beneficial (pp. 51-52).

It would appear from the discussion of the two teachers in this paper that Michelle could be classified as Task Focused, and Lyn as Child Focused. The concept of being Multi-Focused is useful as a model because, while it acknowledges the idiosyncraticity of the student's own approach to a task, it also enables the student to be supported with the loan of the language of the "expert".

This chapter showed both teachers supporting their students when they were having trouble doing tasks independently. Lyn's way of providing support was characterised by the use of mentor support, which may have been her own or that provided by other students. Michelle provided scaffolds which were predominantly in the form of routines or frameworks. The discourse in the two classrooms in this paper has provided insights into the different ways in which each teacher developed mathematical language to support the mathematical concept under discussion.

Summary

Over the last twenty years, teachers have been confronted with various movements including mastery learning, discovery learning, problem-solving, and constructivism. Teachers, therefore, are faced with the dilemma of deciding what is most appropriate for them as teachers, and for their classroom. At one end of the spectrum teachers are to leave children to deduce the strategies and mathematical concepts for themselves, at the other end concepts and strategies are to be taught by direct instruction followed by sustained practice by the children. Scaffolding, however, would appear to utilise components from each. Lemke (1988) stressed the need for academic success involving the accomplishment of the language of the subject area and therefore this is usefully included in teachers' scaffolding discourse. If scaffolding is to have a place in the mathematics classroom it needs to assist students to deduce strategies and mathematical concepts for themselves, while at the same time enabling their own language to be refined toward developing appropriate communication skills.

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Gambling and Ethnomathematics in Australia

Robert Peard

This project began from an idea from reading Peter Carey's novel *Oscar and Lucinda* in which Carey (1987), writing a story set in early Australia, commented that the colony developed as if it were founded on gambling. I have had an interest in the cultural aspects of mathematics, self-generated mathematics, and the mathematics learned out-of-school for some time. In addition I have carried out research into the teaching and learning of probability and for a long time have had an interest in games of chance and the mathematics of gambling. Thus, it seemed natural for me to combine all of these interests into the one study.

Overview of the Study

The study explored how the social backgrounds of a group of students contributed to their intuitive knowledge in probabilistic reasoning, and influenced their processing of the associated mathematics. A group of Year 11 students who came from families for whom the phenomenon of track gambling formed an important part of their cultural background was identified. Another group consisting of students in the same mathematics course but from families for whom the phenomenon of gambling in any form was totally absent from their social backgrounds was identified.

The research employed a qualitative methodology in which a phenomenographic approach was used to investigate the qualitatively different ways in which individuals within the two groups thought about concepts involving probabilistic reasoning, and processed the related mathematical skills and concepts. The cognitive processes involved in the applications of probabilistic and related mathematical concepts in a variety of both gambling and non-gambling situations were studied in order to determine whether this culturally-based knowledge could be viewed as a type of "ethnomathematics."

Data were obtained through individual structured interviews which enabled patterns of reasoning to be compared and contrasted. Analyses of these data enabled intuitive mathematical understandings possessed by the gamblers not only to be identified but also to be linked with their social backgrounds. Also, differences between how individuals in the two groups processed probabilistic and associated mathematical knowledge were determined. This research complements and extends existing knowledge and theories related to culturally-based mathematical knowledge. Implications for further research, for classroom teaching, and for curriculum development in the study of probability in senior secondary mathematics classes will be made.

Ethnomathematics

The term "ethnomathematics" can be applied to that particular informal, even unconscious mathematics, that is implicit in the everyday activities and ways of thinking of any of the reasonably well-defined cultural or sub-cultural groups represented by students in secondary mathematics classes. For example, children who regularly assist parents who are small business proprietors would be expected to engage frequently in activities involving calculations, classification and measurement which would not be unlike the activities of the students in the present study.

In this sense then, the typical secondary mathematics class represents the coming together of a whole range of different kinds of ethnomathematics: there is the ethnomathematics of sport; of music; of small business operation; of track gambling activities; and so on. The problem for the curriculum developer, text book writer, teacher and examiner is how best to identify and take advantage of the intuitive, unconscious understandings brought in to the learning environment by students from backgrounds that include these interests. Ethnomathematics has been defined by D'Ambrosio (1985) as:

The mathematics which is practised by identifiable cultural groups, such as national tribal societies, labour groups, children of a certain age bracket, professional classes and so on. (p. 45)

In this study the term was taken to mean the inherent mathematical ideas that arise naturally out of cultural practices and norms and, in particular, the probabilistic skills and concepts that arise from the cultural practices of a segment of the school population whose social background includes gambling. These students might reasonably be expected to bring these skills and concepts with them to the school environment. Borba (1992) sees ethnomathematics as:

A field of knowledge intrinsically linked to a cultural group and its interest, being in this way tightly linked to its reality ... and being expressed by a language, usually different from the ones used by mathematicians. (p. 134)

The members of one of the two cultural subgroups of this study are identified through a common social interest in the field of track betting. This common interest includes betting on horse racing, dog racing, and trotting events both on and off track. The term "ethnomathematics" is used in the context of the definition given by D'Ambrosio (1985) who argued that its identity

depends largely on focuses of interest, on motivation, and on certain codes and jargons which do not belong to the realm of academic mathematics. (p. 45)

Nunes (1992, p. 557) observed that D'Ambrosio's use of the term ethnomathematics refers to forms of mathematics that vary as a consequence of their being embedded in cultural activities whose purposes are other than the doing of the mathematics. This working definition is in keeping with that employed by other researchers in the field.

Zepp (1989), in noting that the term is difficult to define precisely, suggested that it is useful to consider what the term is not. He suggested that:

It is not a collection of interesting folk games, measuring techniques, or counting systems used by various "primitive" cultures ... nor is it a doctrine which states that differing races have differing mathematical abilities. (p. 211)

Graham (1988), in researching the ethnomathematics of some groups of Aboriginal children in Australia, used the term to refer to "the mathematical understanding that the Aboriginal children bring to the educational encounter ... the mathematical relationships inherent in their own culture" (p. 121). Gerdes (1988) used the same definition to describe the intuitive mathematics of a native culture in a post colonial society. Carraher, Carraher and Schliemann (1985) in Brazil used the term "ethnomathematics" to refer to "the everyday use of mathematics by working youngsters in commercial transactions" (p. 21).

Lampert (1986) and Leinhardt (1988), in separate studies, focused on what students know before instruction as a result of interaction with their social background in an endeavour to determine what "understanding mathematics" means to students who are being taught new aspects of mathematics.

It is well established that many people experience a "psychological blockage" when it comes to learning mathematics. One of the objectives of the present study was to incorporate certain principles arising from the study of ethnomathematics into the curriculum in order to help overcome this "psychological blockage" that is so common in mathematics.

Gambling as Ethnomathematics

Track gambling is an identifiable cultural practice within the Australian social context. When I was a high school teacher I often taught students who were seemingly poor at mathematics, but who could readily perform computations in gambling contexts, particularly those in track gambling contexts. The extensive interest in gambling in Australia and these observations suggested to me as an academic that I should research this more thoroughly. Questions arose such as: Is gambling a form of ethnomathematics within Australian society? What intuitive mathematics do gamblers bring to the learning environment with them and what are the implications of this for those involved in mathematics education? Thus the term "ethnomathematics" is used in this study with some flexibility which is entirely appropriate since, as Bishop (1988, p. 180) has said, "the term ethnomathematics itself is not well defined."

Gambling and Probability in Mathematics Education

The topic of gambling has not been well researched in any academic area, and until recently gambling has not been considered as a suitable topic for academic research. While probabilistic reasoning has recently become a subject of research in mathematics education, much of the earlier research of the understanding of the topic and the processing of the related concepts has been in the domain of

psychology. International research into the topic of gambling has been done mostly by psychologists.

"Chance and Data" incorporates the study of probability and is now an important component of mathematics curriculums at all levels of instruction. The mathematics of gambling is now included in many modern mathematics syllabus documents (See for example *A National Statement on Mathematics for Australian Schools*, Australian Education Council, 1991). There is a clear need for research into probabilistic reasoning and related mathematical concepts which I decided to approach from the following perspectives:

1. The activities of gambling are inherently mathematical in nature.
2. Gambling is widespread within Australian culture.
3. There is an established need for research into ethnomathematics and self-generated mathematics.
4. Probability is now established as a topic in the school curriculum but there is a paucity of research into the probabilistic reasoning of secondary school students.

The Inherent Mathematical Nature of the Activities of Gambling

Many of the activities involved in track gambling are inherently mathematical in nature. These include the calculation of expected returns and winnings at various odds, comparing odds, relating odds and probabilities, calculating numbers of combinations, the concept of mathematical expectation and notions of fair and unfair situations. Epistemological considerations arising from the activities of track betting and other mathematical skills and concepts such as fraction concepts, proportional reasoning and concepts in combinatorics were therefore included in the study. Probabilistic reasoning, including that involved in gambling and games, is part of general mathematical activity, and investigations in mathematics include those of experimental probabilities.

Historical Considerations

An examination of aspects of the historical development of probability theory and probabilistic reasoning is useful in order to understand how it evolved from gambling practices and why it is commonly reported that the teaching and learning of the topic is difficult. Epstein (1977) commented:

Throughout the entire history of man preceding the Renaissance, all efforts towards explaining the phenomena of chance were characterised by comprehensive ignorance of the nature of probability. (p. 92)

According to Epstein, the first reasoned considerations relating to chance came in the sixteenth century. Cardano (1501-1576) is credited with the first attempt to organise the concepts of chance into a cohesive discipline. Before Cardano, the connection between gambling and mathematics was not overtly realised, despite the prevalence of gambling in many societies for many centuries prior to this.

It is known that dice games were played for gaming purposes by the Egyptians as early as 2000 BC. Nevertheless a mathematical theory of probability did not emerge until the seventeenth century in Europe. This theory developed from the need of gamblers to quantify chance occurrences. Gamblers can rightfully claim to

be the godfathers of probability theory, since they are responsible for provoking the stimulating interplay of gambling and mathematics that provided the impetus for the study of probability. For several centuries, games of chance constituted the only concrete field of applications of probabilistic methods and concepts. Fermat and Pascall are together credited with being the first to place the theory of probability within a mathematical framework.

Kendall (1970) noted that with the advent of Christianity came the belief that nothing happens without cause; nothing is random and there is no chance. This view prevailed and hindered the development of a theory of probability despite the noticeable advances in other branches of mathematics. He commented "humanity as a whole has not yet accustomed itself to the idea [of randomness]" (p. 32).

The difficulty in making the connection between mathematics and chance can be readily appreciated when one considers that mathematicians as eminent as Leibniz (1646-1716) incorrectly concluded that the sums of 11 and 12 cast with two dice have equal probabilities. The famous mathematician d'Alembert erroneously assigned a probability of one-third to the probability of getting a head and a tail on the toss of two coins, presumably using an incorrect equally-likely sample space of three possibilities instead of four. d'Alembert also mistakenly thought that the outcomes of tossing a coin three times were different from those obtained in a single toss of three coins simultaneously. In addition to the lack of a mathematical framework on which to base a theory of probability, other notions inhibited understanding in the field. The notions of hidden cause, determinism, miracles and the like have all inhibited the development of the connection between mathematics and chance.

In *A National Statement on Mathematics for Australian Schools* it is noted that there is an increasing emphasis on random models of the world rather than on an event's inevitable consequences that can be described by rules and equations. It is important to know that many events are the result of chance variation rather than deterministic causation so that it not always necessary to look for specific, often spurious, reasons to explain an occurrence.

Gambling is Widespread Within Australian Culture

The phenomenon of track gambling is widespread within Australian culture. Indeed, it can be argued that gambling is related to this culture in a unique way.

Expenditure on Gambling. Evidence of the extent of gambling in Australia is furnished by considering the expenditure on gambling. All figures are of necessity either estimates or approximations and are constantly changing.

Legal Betting. Fairly reliable estimates of the amount spent on this are available from a variety of sources. Haig (1985), has provided much data on the topic. He qualified his estimates with the recognition that difficulties in obtaining accurate estimates relate to "the omission of illegal betting, understatement of bets by licensed bookmakers and expenditure on miscellaneous activities which are not taxed" (p. 77). He has also noted that attempts to make comparisons of expenditure between countries are difficult owing to factors such as "inconsistent definitions and incomplete coverage" (p. 73). Nevertheless, Haig concluded that "the figures

indicate that Australia has the highest level of gambling expenditure per head of population of any country" (p. 74).

Warneminde, writing in the *Weekend Australian*, August 14, 1991, reported that a study by a private data firm, Australian Gambling Statistics, estimated that Australians lost a total of \$4400 million in 1989-90 in legal gambling activities. This figure represents \$2380 for every Australian over the age of 18. He commented:

There is statistical support for the reputation of Australians as a nation of people who like a bet. The evidence shows that we lead the world in our enthusiasm. (p. 81)

Haig (1985) estimated that when the data were broken down into the various forms of gambling, 70 percent of all legal betting was on track events. However, more recent estimates suggest that this proportion is less than 50 percent. This decrease does not represent a dollar decrease in track betting, but can be attributed to the increase in expenditure associated with the introduction of casinos and an increase in the number of poker machines following their recent introduction in many states. Although casino and poker machine betting has continued to gain popularity and increased revenue over the last few years, track gambling expenditure has not decreased.

Illegal Betting. Haig (1985) commented that "most illegal gambling is track betting, and it is likely that illegal gambling is higher in Australia than in other countries because of the relatively greater importance of track betting" (p. 74). Safe (1992) reported that the Queensland Criminal Justice Commission estimated the illegal off-course SP (starting price) market to be worth over \$4 billion a year.

Social Acceptance of Gambling

In addition to statistics relating to expenditure, we find evidence for the widespread occurrence and acceptance of gambling in the quantity of newspaper space concerned with track racing and the time given to the broadcasting of track events and their results on radio and television. There is in Australia an acceptance of gambling as a "respectable" pastime that is not to be found in many other societies. Social scientist, Jan McMillen, of the Queensland University of Technology, cited by Warneminde (1991), observed:

There has not been a well organised moral opposition to gambling in this country ... we don't have the moral hang-ups of the USA where after the excess of the Wild West, it became viewed as a vice and subjected to prohibitions which still exist in all but a few states. (p. 81)

Interest in track events is common in the USA, Britain and Ireland, but expenditure per head of population is much lower than in Australia. Many Asian countries are also tolerant of gambling, but with the exception of Hong Kong, little gambling is related to track events and, unlike Australia, it does not generate large amounts of government revenue.

Gambling is an integral part of Australia's self-image. More recently mathematics educators such as Lovitt and Clarke (1988) have recognised that "gambling is widespread in our community" (p. 75) and have included a simulation of the operation of the TAB betting system in their *Mathematics Curriculum and Teaching Program* materials package.

Ethnomathematics and Self-Generated Mathematics

Bishop (1991) maintained that all mathematical learning takes place in a social setting and that we need to be able to theorise about "interpersonal" as well as "intrapersonal" mathematical learning. He stressed that learning mathematics in a social context cannot be fully interpreted as an "intrapersonal" phenomenon because of the social context in which it occurs. Equally, "interpersonal" or sociological constructs will be inadequate alone since it is always the individual learner who must make sense and meaning of the mathematics.

Glaeser (1983) noted that within modern society the ideas of probability are very common:

When one starts to teach ... this subject ... [students] are certainly not without previous knowledge: ... everybody is familiarised with situations of betting, of drawing lots, or with decisions under uncertainty. (p. 313)

Furthermore, there is much evidence that informal procedures learned outside of school are often extremely effective. Gay and Cole (1967), for example, showed that unschooled Kpelle traders estimated quantities of rice far better than educated Americans. They became convinced that it was necessary to investigate first the "indigenous mathematics," in order to be able to build effective bridges from this "indigenous mathematics" to the mathematics of the school.

Eduardo Luna (in Gerdes, 1988) argued that the practical mathematical knowledge that children acquire outside the school is "repressed" and "confused" in the school. In a similar vein, Carraher and Schliemann (1985) have shown that children who have to make frequent and quite complex computations outside of school did so efficiently in out-of-school contexts, but were not successful with the same type of computation in a classroom context.

The importance of the cultural context in mathematics education has formed a central theme of much recent research. A common element of projects in ethnomathematics and self-generated mathematics is that the legitimization of the learners' experiences is recognised as being of fundamental pedagogical importance.

Ethnomathematics, Constructivism, and Mathematics Education

The present study adopts the constructivist perspective that learners construct or invent knowledge on the basis of what they already know, and that much of what they already know has developed from cognitive interaction with factors in their social background. Other researchers have clearly shown that learners do invent useful strategies to solve novel problems. Constructivism acknowledges the relativistic nature of the constructions and recognises that constructed concepts are valued for how they can be used to deal with problems. Studies researching mathematics used in the workplace (Carraher et al., 1985; Cockcroft, 1982; Lave, 1988; Scribner, 1984) have shown that such mathematics is often idiosyncratic. Cockcroft (1982) referred to "back of envelope" methods as opposed to formal algorithmic methods taught in school. The present research examines

context-specific procedures in probabilistic situations used by the gamblers to identify the context-specific intuitive knowledge required for such computations.

Probability in the Secondary School Curriculum

It was noted by von Glasersfeld (1987) that the nature of probability is pedagogically suited to the contemporary mathematics educator's belief that children are active constructors of their own knowledge. This, he maintained, is due to the experimental nature of the topic and its emphasis on inquiry.

The importance of understanding probabilistic concepts in modern technological societies has been well established for some time now. It has been argued that it is essential that students be taught how to deal realistically with uncertainties otherwise they may respond to probabilistic situations with preconceived notions, emotive judgements and even a lack of awareness that chance effects are operating. Despite the recognition of the importance of probabilistic concepts by mathematics educators, the inclusion of probability into the mathematics curriculum is a relatively recent development.

Mathematics educators such as Watson (1992) have expressed concern that recent initiatives in curriculum development "have been taken without the benefit of previous educational research in Australia on the learning of probability" (p. 1). She stated that:

In Australian school systems teachers are currently implementing the Chance and Data curriculum using the best resources and advice they can get from educators and curriculum planners, all of whom are operating without the luxury of a local research base. (p. 5)

She then argued that since probability is such a relatively new area of the curriculum, research is needed to "provide a fundamental structure" (p. 11) for teaching and learning. She concluded that in the wake of the publication of *A National Statement on Mathematics for Australian Schools* there is "an urgent need for research into the understanding of concepts related to probability" (p. 13). Watson (1992) also made the important point that the probability and statistics component of the mathematics curriculum is one part that is closely related to out-of-school experiences.

The present research includes an examination of probabilistic concepts in real world, out-of-school contexts. An extensive review of the literature revealed that no comparable studies have been reported. With reference to gambling, only studies of adult gamblers by psychologists are reported (Ceci & Leiker, 1986a, 1986b; Kahneman & Tversky, 1982).

The Major Research Questions

Following an extensive review of the literature related to ethnomathematics and probability three major research questions were formulated. These were:

Major Research Question 1

Do Year 11 students of "Mathematics in Society" (a lower stream course in Queensland) whose social background includes extensive familiarity with track gambling (the "gamblers") have different intuitive probabilistic concepts and understandings from students for whom track betting is absent from their family and social background (the "non-gamblers")?

Major Research Question 2

What are the cognitive processes employed in the application of probabilistic and related mathematical concepts in traditional classroom situations and in out-of-school contexts involving track gambling? Do these processes differ between the two groups specified in the first major research question, and, if so, what are the differences in the ways individuals in the groups tend to process these concepts?

Major Research Question 3

Is the knowledge acquired by the "gamblers" as a result of socially induced cognitive interactions in gambling contexts sufficiently pervasive to be regarded as a form of "ethnomathematics"?

The Research Methodology

Since these questions called for research into (a) intuitive concepts and understandings, (b) the ways in which these concepts are applied and processed, and (c) the pervasiveness of such knowledge, it was recognised that the data needed for this study should be mostly qualitative in nature, and therefore it was decided that qualitative research methods would be employed.

The aim of qualitative research in education according to Sherman and Webb (1988) is to understand "experience as nearly as possible as its participants feel it or live it" (p. 7). They claim that the function of qualitative research is "to interpret, or appraise, behaviour in relation to contextual circumstances" (p. 10). The present research examined the mathematical behaviour associated with the processing of probabilistic ideas in the social context of track gambling, and is therefore well suited to the use of qualitative methods. It was recognised from the outset that the chosen qualitative research methodology should permit every effort to be made to preserve advantages arising from high "standards of design, analysis and statistical reliability" (Lamon, 1972, p. 8).

Generally speaking, qualitative research data are not suited to statistical analysis and it was not the intent of this investigation to make inferences regarding the populations from which the samples were selected. Nevertheless care in the identification of the population and the selection of the samples was taken since the study was expected to generate hypotheses which might be tested in later studies for their validity and degree of possible generalisation to the population from which the samples were selected.

The Research Populations

It was shown that there is a large segment of Australian society for whom the phenomenon is part of their social background. It was the Year 11 students from families with this kind of background and who were attending schools in the Brisbane metropolitan region in 1991 who were regarded as forming one of the populations for the research which was carried out. The other population consisted of Year 11 students attending schools in the Brisbane metropolitan region in 1991 from families for whom such gambling was completely absent from their social backgrounds.

The Research Samples

Two schools were identified in regions in which it could be reasonably expected that there would be an interest in social gambling within many families. Both schools are in the vicinity of racing tracks (horse, dog, and trotting), and student familiarity with track betting was confirmed. A questionnaire was administered to six classes each of approximately 25 students in order to identify students from the required social background.

Gender Balance Within the Samples. In all of the schools used in this study, and in Queensland in general, "Maths in Society" courses contain approximately the same numbers of males as females. Consequently, it was not necessary to incorporate any special techniques in the methodology to effect gender balance of the samples.

Balancing Achievement Levels. It was decided that in the sampling procedure attention would be paid to pupil achievement level. It was deemed essential that this be included in the sampling procedure since otherwise comparisons between gamblers and non-gamblers might be obscured by differing achievement levels. It would be inappropriate, for example, to compare only low achieving gamblers with high achieving non-gamblers.

A deliberate decision was made to select approximately equal numbers of high and low achievers and to omit the middle level. In this way, differences between the levels would be more easily identified. In Queensland schools, all student assessment is school based and teachers are required to keep extensive and up to date records of student achievement. For the purposes of this research, it was decided that the classroom teacher would be asked to classify the student informally from personal knowledge and from teacher records as either a "High achiever" or a "Low achiever."

Criteria for the Selection of "Gamblers." To be eligible for selection as "gamblers," the students had to indicate that they either attended track events themselves or had parents who were "very interested in" track events. Twenty gamblers were selected.

Criteria for the Selection of "Non-Gamblers." To be eligible for selection as "non-gamblers" students had to indicate not only that their parents had "no interest at all" in track gambling but also that they themselves had no interest in any form of gambling, including games of chance involving the playing of cards or the rolling of dice. Since these activities are very common among most student populations it was decided to use a third school in which students would be less

likely to have these characteristics in their background. A Seventh Day Adventist High School was selected in order to obtain this part of the sample of non-gamblers. Although it was not a requirement for selection as a non-gambler to hold negative or hostile opinions towards gambling, there were nevertheless some students from this school from social backgrounds in which this was the case.

Composition of the Samples: A Summary. Ideally, as a result of the above procedures, we would expect two samples of 20 students and five students in each of the eight cells of Table 1. However, it was decided that since the criterion for gamblers as opposed to non-gamblers was the most important criterion in the selection, no prior rigid decision on the numbers in each cell of Table 1 would be made.

Table 1
Composition of Samples

Code	Group of Respondents						
	High Achievement		Low Achievement		Total		Total
	Male	Female	Male	Female	Male	Female	
Gamblers	5	4	6	5	11	9	20
Non-Gamblers	3	7	5	5	8	12	20
Totals	8	11	11	10	19	21	40

Data Gathering

The major research instrument used for data gathering was *the structured clinical interview* (see Romberg & Uprichard, 1977). The flexibility of this was a factor in its selection, since as Ginsburg, Kossan, Schwartz and Swanson (1983) have stated, "it allows the interviewer to present problems and questions in a flexible manner and that this in turn allows for contingencies that may arise" (p. 11).

The use of the interview as an instrument for gathering data in probabilistic reasoning was specifically supported by Shaughnessy (1992) and Scholz (1991) who both reported that much of the research by cognitive psychologists on the acquisition of probabilistic concepts has methodological flaws, because tasks are posed in a multiple-choice, forced-answer format, where subjects' understanding of the tasks and their reasoning processes are not evident. The establishment of rapport with the student is another feature of the interview that Ginsburg et al. consider to be important. This is necessary in order to create an interpersonal relationship of trust in which the interviewer presents a non-judgemental and supportive attitude.

The qualitative research methodology of the present study incorporated some of the techniques that Marton (1988, p. 154) described as "phenomenography." Marton noted that "interviewing has been the primary method of phenomenographic data collection" and that open-ended interview questions are particularly suited to situations in which the subjects need to be able to "choose the dimensions of the questions they want to answer." In the present research, this is done in order to determine "the qualitatively different ways in which people experience or conceptualise specific phenomenon" (p. 154).

Construction of the Major Research Instrument

In order to study the broad questions formulated, a large set of specific interview questions was constructed. Each of the written questions was administered to interviewees one at a time, with discussion and probing of each. The students were encouraged to respond using whatever technique they chose; written pencil-paper, using a calculator, mental computation with verbal response, or a combination of techniques. Examples of questions asked and the student responses are given in the Results and Analysis.

Data Analysis

Responses to all items gathered were categorised according to the answer given, the technique of computation, the reasoning employed, or certain other identified criteria. Categories identified in the literature such as "incorrect additive technique," "correct functional technique," and any other identifiable categories of response were recorded in each student's file. Where necessary, special categories were constructed by the author. These responses were coded numerically and recorded on a spreadsheet.

It was not the intent to attempt to analyse all the data coded on the spreadsheet. Rather the use of the spreadsheet in data analysis was to enable the selection of categories quickly and easily for comparison, for contrasting, and for pattern identification. All major analyses of data proceeded from examination of the raw data and the students' files. Although responses were coded numerically for ease of identification, all data were qualitative in nature so no scores or summations of marks were used.

Results and Analysis

Intuitive Probabilistic Concepts and Understandings

The first major research question was concerned with Intuitive Probabilistic Concepts and Understandings and data relating to these were obtained from responses to various questions. One important example is *Question 1(b)*. Table 2 illustrates the coding of the responses to this item for analysis.

Question 1(b)

In each of the following situations, how much can be won on a track bet if:

- (i) \$10 is bet at odds of 9:2
- (ii) \$9 is bet at odds of 3:2
- (iii) \$5 is bet at odds of 7:4

Table 2
Coding and Results; Question 1(b), (iii)

Code	Group of Respondent				Total
	G-LA	G-HA	NG-HA	NG-LA	
1. Correct Multiplicative	4	4	7	2	17
2. Correct Scalar Additive	0	2	1	0	3
3. Incorrect Traditional/Functional	0	1	0	3	4
4. Double/Halve with Approximation	1	0	0	0	1
5. Incorrect Additive	1	1	1	4	7
6. Scalar Additive Approximation	5	1	1	1	8
Totals:	11	9	10	10	40

Analysis of data from Question 1(b), (iii). There are 20 "exact" correct responses, of which 10 were from gamblers, and a further nine computational estimations resulting in a close approximation to the correct answer, of which seven were from gamblers. Thus the overall performance of the gamblers on this item was noticeably better than the non-gamblers. Furthermore, eight of the gamblers used a scalar additive technique which was used by only three of the non-gamblers. The reasoning employed by the gamblers in the approximation technique tended to be along the following lines:

7:4 means bet 4 and get 7; So if I bet 5, I must get more than 7;

To figure out how much more: $5 - 4 = 1$;

If 4 gets 7, then this extra 1 gets me more than 1 but less than 2.

[This extra "scaled" amount is estimated then added, hence the terminology employed]

So 5 gets me more than 8 but less than 9—closer to 9 than 8, about (\$8.50 or \$8.75 or \$8.80).

Intuitive Language

The gamblers, in general, demonstrated a greater use of the above unitary strategy. When using either a unitary strategy, a scalar additive approach, or computational estimation strategies, the gamblers tended to use language in different ways from the non-gamblers. The intuitive knowledge possessed by the gamblers identified here clearly derives from their facility with the informal language of track gambling. An important educational issue is the extent to which the informal language and experiences of the students' personal gambling world were linked cognitively to the formal language, symbols and skills associated with school mathematics (Ellerton & Clements, 1991).

Other instances of intuitive knowledge were observed related to the comparison of "odds" and the relationship between "odds" and probabilities. This was demonstrated in responses to Questions 4 and 5.

Question 4

In each of the following track betting situations, which is the better of the two odds? That is to say which gives the greater return per \$1 bet, or which is the greater ratio?

- (a) 2:1 or 3:1
- (b) 5:1 or 5:2
- (c) 4:3 or 9:7

Table 3

Coding and Results; Question 4(c)

Code	Group of Respondent				
	G-LA	G-HA	NG-HA	NG-LA	Total
1. Correct Traditional	4	5	3	4	16
2. Constructs Algorithm;					
Less Outlay for \$2 Win	3	2	0	0	5
3. Scalar Additive with Approx.	3	1	0	0	4
4. Incorrect/Not Applicable	1	1	7	6	15
*Total Correct: 25 (18 Gamblers, 11 High Achievers)					

Analysis of data from Question 4. The results to this question are quite striking. Nineteen of the 20 gamblers were able to answer parts (a) and (b), and an astonishing 18 of these students were able to make a correct comparison in part (c) using some technique. By comparison, only seven of the non-gamblers could answer this correctly. By referring to the spreadsheet data, it can be seen that of the 18 gamblers, 11 did not compare fractions confidently, and nine did not compare probabilities in a non-gambling context. Thus it would appear that nearly all the gamblers demonstrated an intuitive understanding of the terminology and nature of "odds." This knowledge was demonstrated by only seven of the non-gamblers. However, of the 13 non-gamblers who did not make a correct comparison of odds, 11 compared fractions correctly, and six compared probabilities correctly, indicating little intuitive knowledge of the concept of odds.

Furthermore, when the data from Question 4(c) was examined, it was observed that five gamblers constructed a procedure that was *number dependent* (Code 2) in that they used other traditional school strategies for parts (a) and (b); and context dependent in the sense that they did not employ the strategy in other questions requiring proportional reasoning (See further in the Results and Analysis for Research Question 2).

Question 5

In a four horse race the odds for each horse are given as 2:1, 5:3, 5:1, and 25:1. Which horse is thought to be most likely to win?

List the odds in order of least likely to most likely.

Table 4
Coding and Results; Question 5

Code	Group of Respondent				Total
	G-LA	G-HA	NG-HA	NG-LA	
1. Correct Multiplicative	5	7	7	2	21
2. Correct, Intuitive	6	2	0	1	9
3. Unable	0	0	3	6	9
4. Incorrect Additive	0	0	0	1	1
Totals:	11	9	10	10	40
Total Correct:	30	(All 20 Gamblers and 10 Non-Gamblers)			

Analysis of data from Question 5. The gamblers performed noticeably better than the non-gamblers (20 correct compared with 10). This may be attributed to the gamblers familiarity with the context. In those responses classified as "intuitive" (Code 2) the students recognised that 2:1 is greater than 5:3 but were unable to give a mathematically correct reason. Probing questions were then asked. A typical response of the eight gamblers in this category was "2:1 are higher odds because you get more than at 5:3." With further probing, using questions such as "Why do you get more?", this response could, in some instances, have been reclassified as unitary or equivalent proportional. Unitary reasoning involved the recognition that 5:3 returned "less than 2" for the one bet, while equivalent proportional reasoning involved explaining that "at 2:1, a bet of three would win more than 5."

Thus, familiarity with the context would appear to have resulted in the intuitive understanding of some of the gamblers. Further probing questions showed that the gamblers tended to have an intuitive understanding of the relationship between "odds" and probability. Students were not asked to convert odds to numerical probabilities, but were required to demonstrate a knowledge of the relationship that the greater the odds, the less the probability of winning. While all the gamblers were able to list the odds in the correct order of likelihood, only 13 non-gamblers were able to do this.

Intuitive Understanding of Fairness and Expectation

One of the major findings of the study related to the concept of mathematical expectation. The responses to Question 10 are presented for analysis to illustrate the differences between the gamblers and non-gamblers.

Question 10(a)

Suppose you and I play a game with one of these dice [show single die]. If on a single roll it is three or less you win, if it is more than three I win. If we each bet \$1 and the winner gets the \$2, would this be a fair game?

Results from Question 10(a). Thirty-eight of the forty students recognized this as a fair game. Only two non-gamblers responded that they were unable to answer because they had no basis for making a decision in such a situation.

Question 10(b), (i)

Suppose we change the rules so that I win if it is a three or more. Is this still a fair game?

Results from Question 10(b), (i). 27 recognized the unfairness of this situation and the following extended probe was conducted in the interviews:

Question 10(b), (ii)

Can we make the game fair somehow?

Analysis of data from Question 10(b). Of the 27 who recognized that the game was no longer fair, 19 demonstrated some intuitive knowledge of expectation by recognizing that the game could still be made fair by changing the contributions of the players. These 19 were probed further:

Question 10(b), (iii)

How much should I put in to make the game fair?

Table 5

Coding and Results; Question 10(b), (iii)

Code	Group of Respondent		
	G	NG	Total
1. \$2	8	2	10
2. >\$1, But Not Sure How Much	6	3	9

Analysis of the data from Question 10(b), (iii). Of these 19 students, ten (eight gamblers) were able to recognise that since one probability is twice the other, the returns must be in that ratio, thus demonstrating an intuitive understanding of the basic multiplicative principle underlying mathematical fairness. Question 10(c) further probed the concept of expectation. The understanding of the "multiplicative" nature of the concept of expectation was researched in this section. By asking the probing question that reversed the roles of the players ("you" and "I"), this concept was examined in some detail for the few students who demonstrated such an understanding.

Question 10(c)

[If the answer to Question 10(b) is \$2, continue with further questions]

How much should I put in if I choose the numbers 1 through 5, leaving you just the 6?

[Repeat with drawing cards from a deck]

How much should I put in if I choose spades leaving you the other three suits?

How much if I choose a single card of any suit? How much if I choose a single specific card?

[If correct, repeat the questions reversing the order of "you" and "I"]

Results from Question 10(c). Of the 10 who responded correctly to the first part, six were able to answer all parts correctly.

Analysis of Data Relating to the Concept of Expectation

Table 6 summarises the above results, and also incorporates findings from the preceding analysis of all parts of Question 10, and data from the actual student transcripts.

In the table, students who seem to have no knowledge of mathematical expectation were regarded as being in Category 1. These students were unable to answer Question 10(b) and would tend to reason: "A game can only be fair if each player has the same chance of winning." Two "non-gamblers" admitted to having no basis on which to make decisions of fairness.

Students allocated to Category 2 showed some intuitive knowledge of the use of expectation in the determination of fairness, answered Question 10(b) correctly and were able to recognise that a game could be made fair by varying the amounts paid, at least in simple situations. However, they were unable to answer the more complex questions that followed.

Category 3 students demonstrated a thorough knowledge of the concept of mathematical expectation. This was demonstrated by the interviewees' correct responses to all questions in this section. To qualify for inclusion in this category, the student had to be able to demonstrate that in all of the situations, fairness could be established by each player contributing an amount that is in inverse relation to the probability (the constant product requirement). In addition, the students had to be able to compute the amounts correctly.

Table 6
Evidence of Knowledge of the Concept of Expectation

Category	Group of Respondent				Total
	G-LA	G-HA	NG-HA	NG-LA	
1. No Knowledge Evident	3	4	7	7	21
2. Some Knowledge of Concept	5	3	2	3	13
3. A Thorough Knowledge	3	2	1	0	6

Expectation and school achievement. It can be also be seen from Table 6 that there is no strong relationship between the understanding of mathematical expectation and school achievement.

Expectation and gender. In order to examine the relationship between knowledge of expectation and gender, Table 7 was constructed in which the data suggest that there is no strong relationship between the understanding of mathematical expectation and gender.

Table 7
Separation of the Data Relating to the Knowledge of the Concept of Expectation by Gender

Knowledge	Gender		Total
	Female	Male	
1. None	12	9	21
2. Some	6	7	13
3. Complete	3	3	6
Total	21	19	40

Implications of the Results of the Analysis of Data Relating to Expectation

These results have two important implications. First, the concept of expected return is not part of the regular school curriculum at this level—the students do not use the term “expectation” but construct a procedure that recognises the inverse relationship between probability and return. In this sense they are constructing knowledge in the manner described by the constructivists mentioned in the literature.

Second, since there is a noticeable difference in this conceptual understanding between the two groups relating to gambling background without relating to either school achievement or gender, it may be conjectured that this understanding results from their familiarity with the related concept of “return on track bets at given odds.” Further research may be necessary to test this hypothesis generated. Nevertheless, we may conclude that one field of probabilistic reasoning in which the gamblers demonstrated an intuitive understanding and the non-gamblers did not, was that of the concept of fairness and its relationship to mathematical expectation and expected return.

Absence of Intuition

Generally speaking, students from both groups were able to quantify simple probabilities without difficulty, but the ability to compare probabilities related more to school achievement and to the ability to compare fractions in traditional school contexts, than to gambling background. There appears to be little, if any, difference between the two groups in intuitive understanding of basic simple probability.

Data from the responses to items relating to compound probability pointed to similar conclusions to those reached by Brown, Carpenter, Kouba, Lindquist, Silver and Swafford (1988) that “knowledge of all but the simplest of probabilistic questions is extremely limited” (p. 242). The multiplicative nature of compound probability is not well understood by Year 11 students in any of the contexts, gambling or non-gambling and this inability does not relate noticeably to gambling background or school achievement. As Fischbein, Nello and Marino (1991) commented, there appears to be “no natural intuition for evaluating the probability of a compound event” (p. 534).

Very few students in either groups demonstrated any intuitive understanding of the basic principles of combinatoric knowledge. Although the mathematics of track betting on the “quinella” and the “trifecta” involve combinatoric concepts, the gamblers showed no better intuitive understanding of these concepts than the non-gamblers.

Results and Analysis of Data for the Second Major Research Question

This question was concerned with the differences in the ways individuals in the groups tend to process probabilistic concepts. First, some generalisations can be made regarding overall responses.

Comments on Responses According to Gender. An examination of the data showed that for most items there were no markedly noticeable gender effect on the quality

of responses. This is not unexpected, since, as was noted, there is no gender bias of achievement in this course at any of the schools involved in the study, or in the State in general.

Comments on Responses According to School Achievement. The quality of responses to many items, especially the "traditional" classroom questions, showed a noticeable relationship with school achievement level. As would be expected, high achievers tended to perform better than low achievers.

Triangulation of Results. The research question was concerned with *differences* in ways by which concepts are processed, the language used, and the computational techniques employed. In some questions (Question 1 for example) the numbers and concepts were the same but the context was different. The differences in context were accompanied by differences in the students' language and techniques of computation. These three difference all pointed to the same conclusion, namely that there are differences in the way in which individuals within the two groups process mathematical concepts. This is illustrated by the triangulation of the results of Question 1 (Figure 1) involving the concept of proportional reasoning.

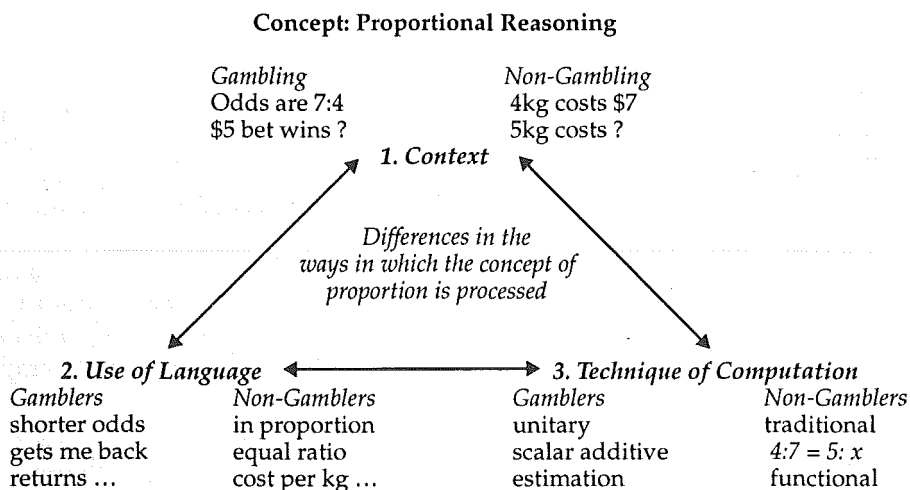


Figure 1: Triangulation of Results

A different technique was also noted in the responses of five gamblers to Question 4(c). The reasoning employed was along these lines:

4:3 is the same as 8:6; This means bet 6 win 8, 2 more than you bet; 9:7 means bet 7 win 9, again 2 more than you bet. It is better to win the same amount for a lower outlay, so 4:3 are the better odds.

Mathematically, this reasoning is correct although it is not referred to in the earlier extensive review of the literature on strategies employed in proportional reasoning. Its use by the gamblers in the present study was highly context

dependent, in the sense that none of the gamblers who used it in this context did so in any other contexts. Its use was also number dependent in the sense that none of the gamblers used it in parts (a) or (b).

Results and Analysis of Data for the Third Major Research Question

An answer to the third major research question was formulated by the synthesis of data from the first two major research questions. This involved the recognition of the different practices, codes, jargons and styles of reasoning employed by the gamblers in the context of track gambling. Although this language, methods of computation and ways of thinking associated with track gambling are learned in the sense that they are acquired through participation in a particular sub-culture, they are not formally studied. Often their application is more an unconscious than a conscious act.

The data support the conclusion that a form of ethnomathematics does exist. However, the practices and styles of reasoning were employed by only some individuals of the group and were not universal among the group members. Some of these individuals used number dependent strategies, such as a scalar additive strategy, effectively in both gambling and non-gambling contexts but these practices and styles of reasoning did not, in general, transfer to traditional school contexts, and therefore were of limited applicability. Nevertheless, it is contended that the practices and styles of reasoning do, in fact, constitute a type of ethnomathematics as defined by D'Ambrosio (1985).

Conclusions and Recommendations

Expectation and Fairness

The possession of intuitive understandings of the important concept of mathematical expectation by a majority of the gamblers gave rise to one of the most noticeable differences between the two groups, and represents one of the major findings of the study. It is recommended that further research to test the following hypothesis be carried out:

Familiarity with track betting situations which involve the computation of the return from placing bets at various odds leads to the development of intuitive understandings of the concepts of mathematical expectation and expected return.

Implications. The results of the present research regarding the concepts of mathematical expectation and fairness have particular implication for curriculum development. These concepts feature prominently both in the Queensland senior mathematics syllabus and in the *National Statement*. In the past, the inclusion of these concepts has been confined to the more academic courses in senior secondary mathematics. This study has shown that these relatively sophisticated mathematical concepts can be understood, at least in part, by a good proportion of non-academic students.

This finding provides strong support for the inclusion of these concepts in the mathematical education of all students, regardless of social background or prior

school achievement in mathematics. Further, it is possible that the use of gambling contexts to introduce and develop these concepts could prove to be highly effective. This is especially likely to be the case if the concepts are introduced in practical ways which link school mathematical concepts with meaningful practical activities.

The Use of Language

The use of a "gambling" language was another of the major differences between the two groups and represents another major finding of the study. However, strong links between this informal mathematics and the formal mathematics of the classroom were not necessarily present. The need to develop curricula aimed at fostering such links, and to provide appropriate professional development programmes aimed at helping teachers to create learning environments likely to assist students to make these links is therefore recommended.

Mental Computation and Estimation

The gamblers employed a number of mental computational techniques effectively in gambling contexts, exhibiting a strong number sense in these contexts. These techniques often incorporated computational estimation and approximation. However, the use of these techniques by the gamblers did not, in general, transfer to traditional non-gambling contexts.

The findings from this study suggest that, provided students can establish links with their own personal worlds, then they are capable carrying out even complicated estimations and approximations. However, the challenge for educators is to develop approaches which are likely to result in students with isolated number sense skills being able to connect these skills with a much broader range of contexts.

Gambling as a Form of Ethnomathematics

It would be reasonable to argue that the special codes, jargons and computational practices of the gamblers do, in fact, constitute a form of ethnomathematics as defined by D'Ambrosio (1985). It has been noted that teachers and educators need to take account of the knowledge which children bring to the school environment as a result of their cultural backgrounds in general, and their out-of-school experiences in particular. The findings of this study have shown that in Australian society there are cultural practices in gambling that generate the development of intuitive knowledge in the area of probabilistic reasoning. These findings support the claim by Clements (1988, p. 5) that "often in Australia there are unique factors influencing how children learn mathematics."

D'Ambrosio (1985) has noted that there is a need to incorporate features arising from the study of ethnomathematics into the curriculum in order to avoid the "psychological blockade" that is so common in school mathematics for many students. One of the features identified in the present study that might be

effectively used for this purpose is the informal mathematical language used by the gamblers, which seems to assist the communication and understanding of informal and formal probabilistic concepts.

Teachers need to explore the relationship between the students' perceptions of probability based on their informal out-of-school experiences, curriculum design, and teaching methodology, with the aim being to maximise student learning. Towards this end, Ellerton and Clements (1991) call for teachers to provide classroom experiences that assist students to make the cognitive links between school mathematics and the personal worlds of the learners.

It is not easy for teachers to develop an ethnomathematical approach to their teaching, however, because they themselves are the product of a mathematics education subculture which encourages them to emphasise isolated mathematical facts, skills, and outcomes. Ellerton and Clements (1991) have noted that

often teachers think they are providing learning environments that encourage students to construct meaning in mathematics, when in fact, the children are, ever so subtly, being required to respond to teacher initiatives most of the time and are being led along comparatively rigid paths towards preset goals. (p. 15)

The present research has shown that for many students the informal mathematics associated with gambling is part of their personal worlds, but rarely do curriculum developers and teachers take account of this. Mathematics education research is nothing more than an academic exercise unless mathematics educators take deliberate steps to make the results of such research known to those having teaching and curriculum development responsibilities for school mathematics in Australia. The challenge to mathematics educators suggested by this study is to make its findings known to curriculum developers and teachers so that ultimately more students will be able to link their personal worlds with the school mathematics they are expected to learn.

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The Role of Imagery in Representing Number

Joanne Mulligan and Noel Thomas

Children's images of mathematics can reveal a wealth of evidence about the ways in which children use representations in developing mathematical concepts. Mathematical concepts and images are difficult to separate, as they are inextricably linked with each other and interact in different ways in learning and understanding. We also know that the learning process does not happen in isolation; children construct mathematical ideas by interacting with others, and by establishing shared meanings for mathematics.

Mathematical representations may take a variety of forms: auditory, verbal, physical, kinaesthetic, visualised, pictorial or notational. Children may model and draw mathematical situations, justify and explain them, act them out or represent them on a calculator or computer screen. In a sense, all representations are mathematical symbols, but some are more abstract than others. Children are exposed to a range of increasingly complex mathematical representations in their everyday experience, often created by adults to articulate mathematical ideas in formal or novel ways.

In this chapter we address the question of how children's images influence the way they represent early number concepts and relationships, and how representational systems are constructed by the child over time. It has not yet been established exactly how children use their representations in building mathematical concepts. If children's representations of their computational strategies are a reflection of their developing concepts, or if their representations actually enhance the development of mathematical concepts and processes, then their representations may provide valuable evidence of how conceptual understanding grows. We attempt to throw light on this question by examining some theoretical approaches, and by drawing upon several related research studies on early number learning that analyse children's imagery as various forms of representation.

Theoretical Perspectives: Imagery and Representational Systems

A number of theoretical approaches are concerned with the way imagery is central to the construction of internal and external representational systems. The early work of Piaget and Inhelder (1967) provided a basis for the direction of research that followed on children's representations of mathematics. They inferred that some kind of internal representation of spatial concepts was necessary in

reference to children's representations (or visualisations), involving the evocation of objects in their absence. These representations were considered to be constructed through progressive organisation of motor and internal actions, resulting in the production of operational systems. The early work of Piaget and Inhelder considered the importance of images as internal, holistic representations of actions which could be inspected and transformed.

Kosslyn (1983) examined the mental processes that were concerned with the construction of the child's internal representation systems. Kosslyn defined four types of image processing: generating an image, inspecting an image to answer questions about it, transforming and operating on an image, and maintaining an image in the service of other mental operations. One of the difficulties is to determine how accurately these internal representations can be inferred from external representations that the child might produce under varying conditions.

Dorfler (1991) also considered the role of imagery in doing mathematics. He believes that an important component of mathematics education is for the learner to be encouraged to make external, his or her internal images. Meaning, in mathematics can be induced by concrete "mental images," and protocoling one's own actions in specific situations can generate a representation of the concept.

Goldin (1992) distinguished cognitive representational systems internal to problem solvers (a theoretical construct to describe the child's inner cognitive processing) from external (task variables and task structures). We draw the distinction between external representations (a structured environment with which the child is interacting that may include, for example, actual physical objects to manipulate and actions in response to that environment), and internal imagistic representations (a theoretical construct to describe the child's inner cognitive processing).

In this model many internal systems of representation, of five different kinds, may develop simultaneously. Thomas, Mulligan and Goldin (1994, 1996) and Thomas and Mulligan (1995), focused on three of the five types of internal representational systems discussed by Goldin: (a) verbal/syntactic systems (using mathematical vocabulary, developing precision of language, self-reflective descriptions); (b) imagistic systems (non-verbal, non-notational representations, e.g. visual or kinaesthetic); and (c) formal notational systems (using notation, relating notation to conceptual understanding, creating new notations).

These internal systems are thought to develop over time through three main stages: (i) an inventive/semiotic stage, in which characters and configurations in a new system are first given meaning in relation to previously-constructed representations; (ii) a structural development stage, where the previously existing system functions as a kind of template on which the new system is modeled; and (iii) an autonomous stage, where the new system of representation functions independently of its precursor, and can assume new meanings in new contexts.

Pirie and Kieren (1992) have proposed a model of the growth of mathematical understanding that reflects a notion of personal building and reorganisation of one's own knowledge structures. The model involves eight potential levels or modes of understanding for a specific person with respect to a particular topic. It attempts to link imagery with the development of understanding. Any level of understanding has embedded in it all other more inner levels of understanding.

Growth of understanding is described as a dynamic organising process whereby extending knowledge involves both abstracting one's understanding to a new outer level and folding back to recursively reconstruct one's inner level knowledge. From a state of "primitive knowing," levels of understanding progress through "image making," "image having," "property noticing," "formalising," "observing," "structuring" and "inventising." At the level of "image having" the child is able to manipulate and use the image in mathematical thinking. "Formalising" entails consciously thinking about the properties of images, whereas "structuring" means being aware of one's formal observations in terms of a logical structure. This relates to Goldin's notion of "imagistic systems" and the phase of "structural development."

Pirie and Keiren's model brings together a number of contemporary views of how children think about and represent their mathematical ideas. The child's use of imagery is seen as central to the way in which children develop mathematical concepts. There seems to be growing research interest in the modes in which children think and represent concepts, rather than developmental stages that they progress through. However, there are still relatively few studies that have investigated and described explicitly the way imagery influences children's development of mathematical concepts.

Representations and Conceptual Development of Number

A broad conceptual basis for number concepts and arithmetic operations develops fundamentally from the child's experiences over a period of time with a range of problem-solving situations, and from establishing relationships between these situations (Davis, 1992; Hiebert & Carpenter, 1992). We now have an extensive body of research indicating that even young children are able to construct number concepts and solve arithmetic problems by using representations in physical, pictorial, iconic or notational forms. Recent studies analysing the growth of number concepts and processes, indicate that children's representations of problem-solving situations are closely linked to their conceptual understanding and the way they construct mathematical relationships (Davis, 1992; Davis, Maher, & Noddings, 1990; Goldin & Herscovics, 1991; Hiebert & Wearne, 1992; Maher, Martino & Alston, 1993). The child uses existing mental representations, and may be required to extend these or construct new representations in order to gain a solution (Davis, 1984). This may promote the development of new mathematical ideas. Children's representations also reveal much about the idiosyncratic and creative ways in which they structure mathematical relationships (Maher, Davis & Alston, 1991; Thomas et al., 1994; Thomas & Mulligan, 1995).

The Role of Imagery in Representation of Number

The role of imagery in the representation of mathematical ideas and in solving mathematical problems has been described by a number of researchers (Bishop, 1989). Some researchers have differentiated between the way students solve mathematical problems using visual or non-visual strategies. Lowrie (1994) suggests that preference for solving mathematical problems visually or

non-visually should not be attributed to "learning style" but to other factors including problem-type, complexity, novelty, or understanding of the problem situation. Preference for solving problems is related to the individual problem encountered and not on ability or learning style.

Hershkowitz and Markovitz (1992) emphasised the importance of visualisation of mathematical concepts and the development of advanced visual thinking. Hershkowitz (1993) and Bobis (1993) investigated the role of visualisation in estimation of number. Hershkowitz showed that visual imagery played a vital role for nine-year-olds in doing numerical tasks, especially in problem solving. Bobis (1993) found that with practice, Kindergarten children were able to use visualising strategies to mentally combine and separate patterns. The children developed subitising skills and started to relate number patterns mentally as they enhanced part-part and part-whole relations. Ten-frame imagery was found to be a useful referent for children in their visualisation of number.

Recent work (Brown & Presmeg, 1993; Brown & Wheatley, 1990; Presmeg, 1986; 1992) in which individual students' thinking was probed in clinical interviews, indicated that students use imagery in the construction of mathematical meaning. Brown and Presmeg (1993) asserted that learning frequently involved the use of imagery although sometimes it might be very abstract and vague forms of imagery. Presmeg (1986) identified five types of visual imagery used by students: (i) concrete, pictorial imagery (pictures in the mind); (ii) pattern imagery (pure relationships depicted in a visual-spatial scheme); (iii) memory images to recall information; (iv) kinaesthetic imagery (involving muscular activity, e.g., fingers "walking"; and (v) dynamic (moving) imagery involving the transformation of concrete visual images.

Recent findings (Brown & Presmeg, 1993) revealed wide differences in the types and facility of imagery used by students in problem solving. All students in the study of seven fifth-grade and six eleventh-grade students used some type of imagery to solve mathematical tasks. Students with a greater relational understanding of mathematics tended to use more abstract forms of imagery such as dynamic and pattern imagery while students with less relational understanding tended to rely on concrete, kinaesthetic and memory images.

Reynolds and Wheatley (1994) reported that fourth/fifth graders used recording to help symbolise mathematical constructions, and children's reflections on these symbolisations elaborated their mental schemes. They found that images were often not well developed but were constructed during reflection on an activity. They suggest that children benefit from these ways of externalising their own meaningful constructions, rather than having the symbolisations of others imposed on them.

Research on Representations of the Number System

Research groups focusing on children's counting strategies (Kamii, 1989; Steffe, Cobb & Richards, 1988; Wright, 1991) and conceptual development of numeration (Bednarz & Janvier, 1988; Cobb & Wheatley, 1988; Denvir & Brown, 1986a, 1986b; Fuson, 1990; Hiebert & Wearne, 1992; Ross, 1990) have highlighted children's

construction of representations of the number system. However, we cannot assume that all children develop uniform understanding of the Hindu-Arabic base 10 number system. Lean's work (1994) suggests that there are thousands of number systems, and that children's development of number concepts must clearly be linked to culture and language.

Rubin and Russell (1992) asserted that children's counting, grouping, estimating and notating skills are essential elements in developing representations of the number system. They described these elements in terms of "landmarks in the number system." These landmarks appear to be related to additive structure, multiplicative structure, the generation and analysis of mathematical patterns and mathematical definitions. Rubin and Russell also suggested that people who are adept with number operations e.g. computing, comparing, and estimating, have a non-uniform view of the whole number system. It is reasonable then to predict that children will develop idiosyncratic constructions of the number system.

Personal visuo-spatial representations of number (number-forms) were described long ago by Galton (1880). Seron, Pesenti, Noel, Deloche and Cornet (1992) studied adults' representations of number claiming that "some automatically 'see' the numbers they are confronted with in a precise location in a structured mental space, others 'associate' specific colours with given numbers" (p. 164). A wide range of representations were found categorised into two main types: continuous lines, scales, grids (coded as number forms), and coloured codes. Associated images and simple analogical representations were also observed. Most subjects asserted that their images of number emerged between five and eight years of age, or that their representations were a direct result of teaching. They concluded that number-forms are used to code the number sequence, and that the function of this phenomenon should be examined in number and calculation processing.

We will now draw upon some related studies investigating children's representations of the number system.

Structural Development of Numerical Representations

Numeration involves the development of an increasingly sophisticated counting scheme and system of notation for recording the numbers generated. Initially a system of units, the counting scheme must drive the representation of assemblages partitioned into groupings of ten. The "ten," while retaining its signification of ten units, becomes itself an iterable unit. This in turn may be operated on by the relation "form a group of ten," to construct the new unit of "hundred." Arrays provide one conceptualisation of the multiplicative process, and can illustrate recursively the relations "multiply by 10," "multiply by 100," etc., in the numeration system.

Thomas (1992) reported the wide variety of mental pictures of the number sequence 1 to 100 that were used by 40 children in Grades K-4. Although some aspects of the structure of number system appeared in the imagery of Grade 2 children, most Grade 4 children still did not possess the structural flexibility with number to successfully mentally manipulate 2-digit numbers.

In a cross-sectional study of 166 children (K-6) and 79 high ability children (Grades 3-6) it was found that the children's internal representations of numbers were highly imagistic, and that their imagistic configurations embodied structural development of the number system to widely varying extents, and often in unconventional ways (Thomas et al., 1994). Children's active processing of internal images were found to be static or dynamic in nature: static meaning a fixed representation, and dynamic as a representation that is changing and moving. In the cross-sectional sample, 3 percent of the children displayed dynamic images of the number sequence whereas 10 percent of the high ability children used dynamic images.

An exploratory study of 77 high ability Grade 5 and Grade 6 children investigated links between their understanding of the numeration system and their representations of the counting sequence 1-100 (Thomas & Mulligan, 1995). Analysis of children's explanations, and pictorial and notational recordings of the numbers 1-100 revealed three dimensions of external representation: (i) pictorial, ikonic, or notational characteristics, (ii) evidence of creative structural development of the number system, and, (iii) evidence for the static or dynamic nature of the internal representation. Children used a wide variety of internal images of which 30 percent used dynamic internal representations. Children with a high level of understanding of the numeration system showed evidence of both structure and dynamic imagery in their representations.

Figures 1 to 6 show some examples of children's representations of the number sequence (Thomas et al., 1994; Thomas & Mulligan, 1995). The children were asked to close their eyes and to imagine the numbers from one to one hundred. Then they were asked to draw pictures that they saw in their minds. They were also asked to explain the image and their drawing.

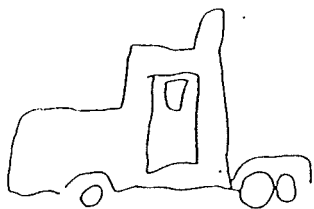


Figure 1: Anthony (Grade 1)

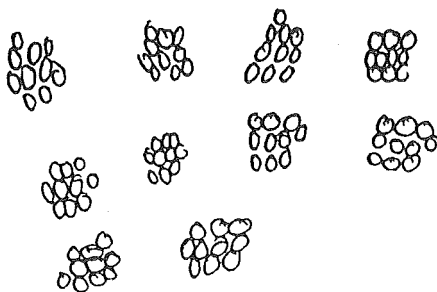


Figure 2: Kimberley (Grade 2)

Anthony (Figure 1) drew a picture of a truck and explained the image as "cause my Dad's truck does a hundred." This external representation was pictorial and we infer an inventive semiotic internal representation relating the truck-image to speed. Kimberley (Figure 2) and Mellissa (Figure 3) produced ikonic representations of the number sequence. Kimberley's recording was of ten groups of ten circles but she could not identify the structure explaining her drawing as "just circles." Mellissa gave evidence of a highly structured imagistic internal representation for the developing numeration system with her drawing of ten, ten rods.

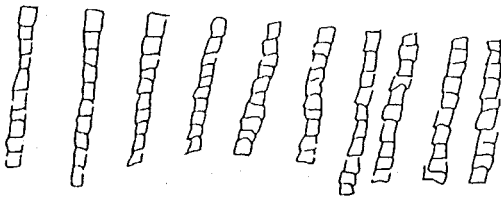


Figure 3: Mellissa (Grade 2)

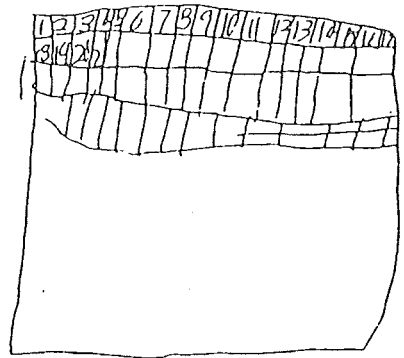


Figure 4: Robert (Grade 2)

Robert (Figure 4) drew a square and subdivided rows of separate squares, each square not being aligned to adjacent squares, and then recorded numerals for the numbers in squares, 1 to 17 being in the first row. This partial array displays an emerging notational structure, but Robert showed evidence of further difficulty with using ten as an iterable unit, saying "you just put the numbers in the boxes as far as you can go ... and you count in ones." Jane (Figure 5) and David (Figure 6) both recorded the number sequence using conventional notation but in highly creative ways. Jane explained that she saw the numbers moving in a spiral formation "going on forever" and it is important to note the structure of segments of number strings in tens (e.g., 71-80, 81-90) that she used. David described numerals flashing one at a time, multiple counting in fives up to 100.

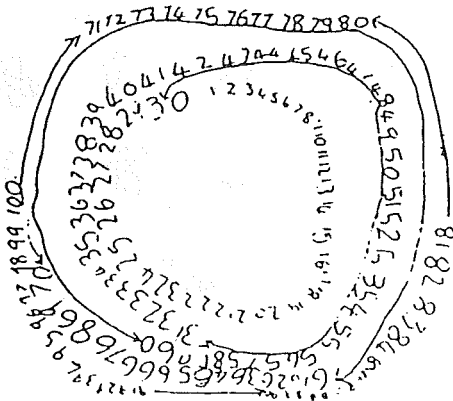


Figure 5: Jane (Grade 1)

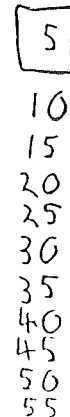


Figure 6: David (Grade 4)

It was inferred that each representation is closely linked to their conceptualisation of structure of the number system. Anthony has not developed structure or a notion of number sequence as yet, but has a sense of the importance and size of 100. Anthony, Jane and David showed highly creative imagery that was not related to their conventional experiences in the classroom. Kimberley, Mellissa and Robert produced images that reflected aspects of classroom experiences. We inferred from Robert's recording that his conceptualisation of the number sequence was linear

although his external representation showed a partial array. He had some sense of the structure of an array being involved but this was not related with organising numbers in tens.

In an action research project involving 140 children from Grades K to 6, Mee (1995) examined children's images of numbers 1-100 by adapting tasks used by Thomas and Mulligan (1995). Figure 7 shows that Oliver produced a detailed drawing of a butterfly with wings partitioned into 100 segments. Although Oliver had a basic understanding of place value of three digit numbers his images of number were dynamic and visualised as real and moving things. Although it appears that this drawing has little structure, Oliver describes it as partitioning into 100 parts to represent hundredths.



Figure 7: Oliver (Grade 2)

Jessica (Figure 8) drew a rectangular grid and subdivided it into 8 columns and 13 rows. Although she wrote the 1-100 counting sequence correctly, she did not use the hundreds chart to represent the structure and pattern of tens. From Jessica's explanation it seems that she used the 100 chart to simply record the counting numbers 1 to 100 without identifying the purpose of using the rows and columns of tens. Her responses to other counting tasks showed that she was unable to count in tens from various starting points, and had poor understanding of place value of three-digit numbers.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64
65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88
89	90	91	92	93	94	95	96
97	98	99	100				

Figure 8: Jessica (Grade 5)

In comparison, further examples from the studies by Thomas and Mulligan show more sophisticated notions of numeration. Cassie (Figure 9) drew an array with the number sequence in rows of ten. She could describe the notational representation as, "a hundred is ten rows of ten." We infer that Cassie's internal

representation involves both the notion of sequence and the idea of groupings by ten, including iteration of that idea relating to the notational system. Edward (Fig. 10) also showed an array structure in his spontaneous imagery for the numbers 1 to 100. When Edward was further asked to show the patterns of ten in the numbers, he too described one hundred as ten tens.

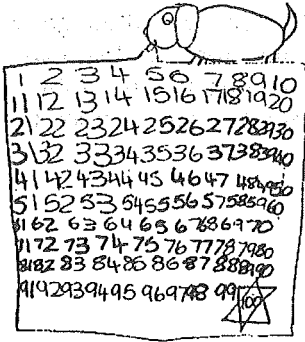


Figure 9: Cassie (Grade 4)

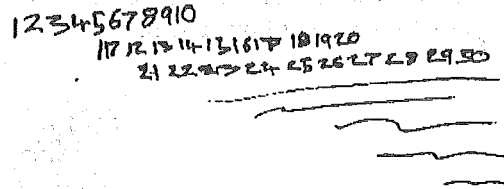


Figure 10: Edward (Grade 6)

From the children's performance on other tasks in the study, there is evidence that Cassie and Edward are able to interpret numerical representations in a variety of contexts, so that as structured systems of internal, cognitive representation, they can reasonably be considered to have reached an autonomous stage of development.

A study of Kindergarten children's understanding of money revealed some very powerful images of magnitude and number. A large proportion of the children were from backgrounds other than English. Goh (1995) asked children to design and draw their own money in notes and coins. Figure 11 shows Shauna's depiction of coins, 5c, 10c, 20c, 50c, 100c, 200c, 300c and 1000c and notes ranging in value from 500 to unrealistic extension of zeros. Her idea was that she would "like to have more money" and that "each note could be worth lots of money," rather than needing more notes of lesser value. Although there is no understanding of the structure she has created, she shows an idiosyncratic notion that more and more zeros will make a quantity larger. In contrast, Figure 12 shows that Fatma used Arabic symbols to create her own notes and coins, using zero as a reversal. The use of the point bears no relation to the use of decimals.

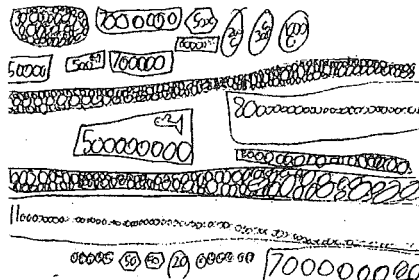


Figure 11: Shauna (Grade 1)

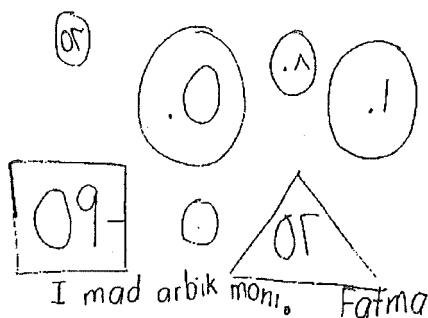


Figure 12: Fatma (Grade 1)

Images of Multiplication and Division Processes

The development of multiplicative reasoning is complex and inextricably linked with children's counting, grouping, partitioning, ratio, additive and subtractive processes. A true understanding of multiplication and division requires the ability to structure, mentally, representations of multiplication and division. For example, consider the classroom example in Figure 13, showing Jane's spontaneous drawing of her images of "space." At first glance it appears that she has drawn a random arrangement of stars, wheels and circles of varying colours (i.e., there were eight different colours used). Notice how she initially creates three objects, stars and circles, and combines these to form wheels. The system that Jane is representing is actually very complex and shows how she has formed multiplicative relationships. Upon closer examination, it can be seen that Jane has represented a system, stars, circles and wheels, drawn four times each in eight different colours. She calculated that there would be 3×8 colours, then 24×4 as 24 doubled, then redoubled to give 96. The interesting point about this example is that Jane expressed understanding of the multiplicative relationship between the factors that she uses, shape \times colour \times number. Her thinking is clearly about forming combinations and she expressed this as "putting things together as patterns."

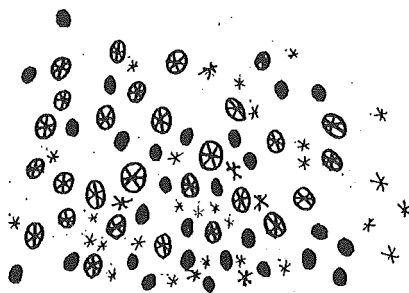


Figure 13: Jane (Grade 3)

Children's internal conceptual structures of multiplication and division, described in earlier studies in terms of intuitive models (Mulligan & Mitchelmore, 1997), were inferred from elements such as grouping, partitioning, counting and patterning of children's external representations, and explanations of these representations. Representations of multiplication strategies have been classified as: (i) modelling the problem by *direct counting* one by one, without taking into account the structure of the equal sized groups; (ii) *repeated addition* of equivalent groups by rhythmic, skip counting or adding, and (iii) the *operation of multiplication*, represented as a binary operation. Intuitive strategies of division resembled those found for multiplication: (i) a rudimentary level, *direct counting*, where children shared and counted one by one; (ii) *repeated subtraction*, where the child started with the dividend and successively took away the numbers in each group, (iii) *repeated addition*, where the child starts at zero and builds up to the dividend by counting or repeated adding, and (iv) the *operation of multiplication* as the inverse of division.

Mulligan and Mitchelmore (1996) report further, an exploratory study examining how third graders spontaneously represented and calculated solutions to multiplicative word problems as they worked in a mini-classroom situation in which they were required to represent and explain their solution strategies. It was questioned how children might be able to solve multiplicative problems if they were encouraged to represent their solution strategies more explicitly through concrete modelling, pictorial, ikonik and notational recordings, as well as verbal and written explanations.

Children were asked to solve and represent the problems in the mode or modes of their choice, but the researcher also encouraged the following methods of response:

- (i) modelling the problem with counters or fingers;
- (ii) drawing a picture or diagram to represent the solution process;
- (iii) representing the problem or solution using notation or as a number sentence; and
- (iv) explaining how they solved the problem verbally and in writing.

The researcher did not teach any specific problem-solving strategies. It was judged important that the problem-solving process be as natural as possible, so the researcher's main role was to encourage a variety of modes of representation. The researcher questioned the children about the similarities or differences between their representations of the problems. For example, did the child use concrete materials to model each problem, or did they draw pictures of the equal grouping structure of the problem in the same way?

Selected examples of the children's representations of equal grouping, cartesian product, partition, quotient and subgrouping problem types, and their method of solution follow. The way individuals linked ikonik, pictorial and notational attributes within a representation, and displayed calculation strategies reflecting intuitive models of multiplication and division will be highlighted.

Figure 14 shows Michelle's response to an equal grouping problem, drawing of "two tables with three children at each table." She formed equivalent groups but focused on her image of the situation as "4 girls and 2 boys," using unitary counting and addition to calculate an answer. This was meaningful for Michelle, but her internal image and notation did not reflect the structure of multiplication.

For the same problem, Rebecca (Figure 15) calculated $3 + 3 = 6$ and used an inventive pictorial and notational representation of equivalent groups. In this example, Rebecca uses equivalent groups and an intuitive model of repeated addition, even though she uses the multiplication sign. The representation also revealed her visualisation of an algorithm in vertical form. Rebecca explained "that this was the way she remembered doing sums." This exemplifies how Rebecca connected previously organised mathematical representations of addition with newly constructed meaning for multiplication. She was able to generalise this vertical form of notation in her representations of all multiplication problems.

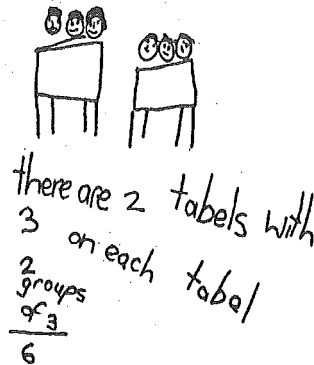
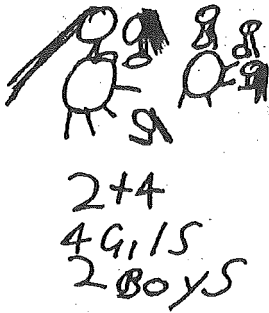


Figure 14: Equivalent groups (Michelle) Figure 15: Equivalent groups (Rebecca)

Figures 16 and 17 show how the children solved a cartesian product problem by drawing the ice creams in order of size and categorised by flavour. Samantha's example was recorded in a vertical array formation symbolised by $4 \times 3 = 12$, whereas Catherine drew the combinations in a linear formation. Catherine had previously used multiplication symbolism in a comparison problem as $4 \times 3 = 12$, but in this situation, chose a multiple pattern of threes to calculate the answer by repeated addition. She was unable to recognise that the problem required the operation of multiplication. Catherine focused on the way the ice creams would look like if they were served "along the counter." She explained that "it was easier to check each one that way."

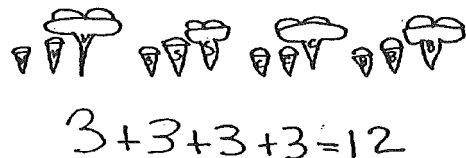
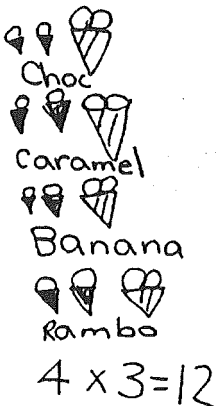


Figure 16: Cartesian product (Samantha)

Figure 17: Cartesian product (Catherine)

The structure of partitive division is exemplified by the representation made by Samantha in Figure 18. Samantha used the more abstract representation of division notation. She began by forming the dividend, which she then partitioned into six equal groups; this was characterised as a repeated subtraction model of division, because she explained a taking-away process in groups rather than counting back directly.

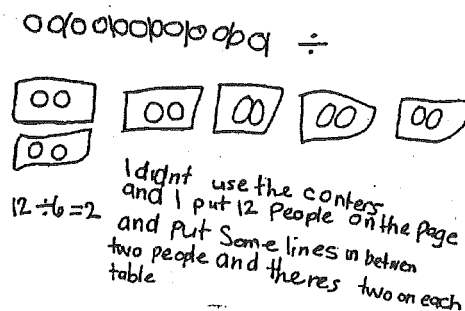


Figure 18: Partitive (Samantha)

Figure 19 shows how Natalie represented a quotition problem and symbolised it as $6 \times 3 = 18$, "6 tables with 3 in each." Again, Natalie uses multiplication notation to represent a multiplicative situation but she relied on direct counting and drawing of each "child" to calculate the answer. Thus, she is not really using multiplication as an operation.

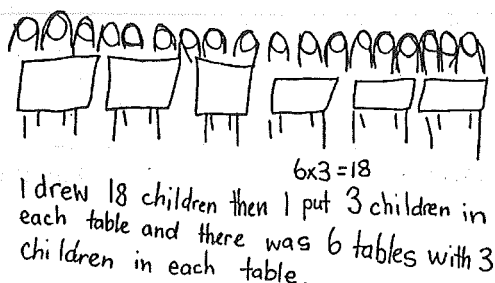


Figure 19: Quotition (Natalie)

The representations used for subgrouping problems also revealed a range of intuitive strategies and invented symbolism which showed variations of the structure of division.

6 Apples 12 people
they get half each

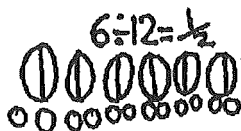


Figure 20: Subgrouping (Bianca)

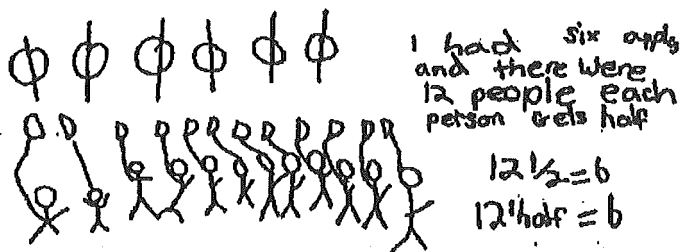


Figure 21: Subgrouping (Catherine)

Figure 20 shows how Bianca represented the apples "halved" and matched one person to each half accordingly. She visualised and explained the halves as separate for each person without drawing them again. As well, she correctly symbolised the process $6/12 = 1/2$ even though she had not been instructed in the use of the division symbol or related part/whole division situations. Catherine used a similar approach and drew the halves matched to each person. She symbolised the process as "12 halves makes 6", but was not aware of any relation to multiplication. Figure 21 shows her use of invented symbolism to specify that she meant "12 lots of $1/2$ " rather than "12 and one half."

Imagery and the Learning Process

In our studies it appears that the active processing of images plays an important part in the development of the child's understanding of numeration and number operations. To facilitate this, children's mental images should be described, drawn, compared and discussed. As their internal structures are

developing, the children's external representations, both static and dynamic, may not correspond to conventional mathematics, or be uniform in nature from one child to the next. They should be expected to reflect each child's unique internal constructions at that time. Such a range of available images is, in our view, healthy; the images are constructed so that an internal representation system that "works" can be built up. Thus the teaching/learning situation needs to provide opportunities for children to develop and represent structurally meaningful mathematics.

Conclusions and Limitations

This chapter has provided many examples of children's representations of number that have been obtained through a range of exploratory and descriptive studies. Although some of the studies have involved large numbers of children they have not been designed or intended as controlled experiments permitting immediate generalization. Our methods of inferring aspects of children's internal representations from their externally produced representations are still exploratory, but current studies will be subject to tests of validity and inter-researcher reliability. Further, data taken in just one or two interviews do not permit us to trace the process of construction of internal representational systems in individual children. Longitudinal studies are needed to trace the development of numerical representations.

With these limitations, there is still growing evidence that cognitive representational systems develop over time. We have described behaviours from which representational acts can be inferred: inventive acts of initially assigning imagistic meanings to, or identifying them with mathematical words and symbols; structural developmental acts associated with sequences of numbers, groupings by tens, recursive grouping, equal grouping structures; and autonomous acts in which insightful, mathematical meanings are freely and flexibly found in other systems of representation, distinct from those used initially in constructing the numeration system.

Our current interest lies in analysing children's development of cognitive structures for number (Thomas et al., 1996). The variations we have observed across different children strongly suggest that representational systems are not fully developed at any one time, but are built up over time. Previously developed representations may serve to provide students with a framework (scaffolding, or template) on which new, meaningful representational configurations can be fit (new knowledge), and new cognitive structures built. During the many steps that occur in the structural development stage of numeration and number operations, we believe that the variety and meaningfulness of the images facilitate passage to autonomous representational systems of number that in turn, influence many aspects of mathematics learning. While the representations may be constructed in response to specific tasks, conceptual understanding of number concepts and processes must involve many experiences with the representation of numerical ideas, across many different tasks, with meaningful semantic relationships among them.

A longitudinal study at Macquarie University of 120 Australian children in Grades 2-3, commenced in March 1996, focuses on children's construction of numerical relationships through representations (Mulligan, Mitchelmore, Outhred & Russell, 1997). This study has shed further light on the processes whereby children's internal systems of representation develop, and how such processes can be inferred from task-based interviews with children.

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Investigating the Mathematical Thinking of Young Children: Some Methodological and Theoretical Issues

Agnes Macmillan

Why Investigate Young Children's Mathematical Thinking?

I have been asked to relate the story of the development and implementation of the theoretical and analytical framework of a study I am in the process of reporting. In its broadest terms, the study is looking at how children are affected by the change from a predominantly child-centred informal pre-school environment to a predominantly teacher-directed formal school environment. As a kindergarten and primary school teacher I was both fascinated and perplexed by many aspects of mathematics education. I was curious about what could be done to harness children's natural capacities and interests, and to maintain intellectual rigour, without the over-challenge which curriculum demands seemed to be necessitating. I was very concerned that my investigation of classroom practices should provide a clear and accurate, but encouraging perspective, because I was only too well aware of the enormously difficult conditions under which kindergarten teachers work. They try to carry out the care-giver role of mother or child-centre professional, often without adequate facilities or support staff, and at the same time, they have to fit in with the philosophy of the school. This can be achieved when one has a strong sense of professional autonomy and when the whole school philosophy is in harmony with one's own.

When young children are playing, or engaged in everyday life experiences, they are learning a great deal about themselves and their intellectual, imaginative, physical, social, and emotional worlds. While children need to have a deliberate intentional orientation to learn skills, such as tying shoelaces or learning to use a knife and fork, abstract knowledge develops in mostly unconscious ways within contexts which are highly salient and meaningful to them. They learn how to structure words to form statements or questions, for example, by practising and rehearsing possibilities and having their attempts at meaning making accepted and responded to positively and constructively. Much of this abstract learning develops as a kind of by-product of active engagement, modelling, improvising or imitating ways of doing and saying things.

The extent to which children become aware of the importance of one kind of knowledge as opposed to another is largely conveyed through the range and diversity of experiences they are exposed to, and the way competence or involvement in particular kinds of experience is valued by their adult caregivers. For example, children come to understand the value of literature through

pleasurable, shared experience with appropriate kinds of texts. The disposition to develop a literate identity is being formed unconsciously in context-embedded and enriching experience.

For young children, learning in an informal, unstructured environment, and is very much a matter of learning “to mean” and “to do” in salient, purposeful and relevant ways. With choice, challenge and control being within their own auspices through play, children learn to be responsibly self-regulative, with adults acting as felicitous consultants or co-participants. But when children come to school, they are introduced to a very different environment. They are required to develop conscious orientations about abstract concepts. They are also required to make explicit and public expressions of conceptions and understandings which are still embryonic and therefore delicate and vulnerable in their state of evolution. When children begin a new experience with vague and uncertain expectations, and then have difficulties in understanding what is intended, the kinds of responses they receive from other people become powerful determinants of the feelings and dispositions they develop. The combination of public accountability of understanding, uncertainty about what is expected and intended in traditional types of mathematics tasks, and limited capacities for reasoning in conventionally acceptable ways, weaken children’s naturally positive dispositions towards learning about the quantifying and qualifying aspects of their physical and spatial world.

An investigation of one aspect of the internal functionings of an organism requires knowledge and understanding of all the other internal characteristics of the organism, as well as the external features of the environment affecting it. Life cannot be maintained in a vacuum, and the vulnerability of early childhood accentuates a need to incorporate both wide-angled and close-up views of what is happening. It requires the study of a particular group or groups of children behaving and interacting as they usually do, with the significant other people in their lives, the resources and technologies available to them, and the wider cultural influences affecting their particular purposes and endeavours. That is, it requires an anthropological study, a process of mapping the culture of a setting.

How Does One Map a Culture?

Yes, it is an awesome question. And well you may ask what has anthropology to do with mathematics research? Is that not a job for sociologists or historians? Indeed, this is true, but if we really want to understand the whole picture of what is happening in mathematics classrooms we need to examine all the dynamic relations inherent in the practices and the environment. It is not a task for the faint-hearted followers of rules and structures, but for the impassioned trail-blazing risk-takers who can tolerate uncertainty, and changing but nonetheless rewarding paths of exploration. It is not a matter of climbing the mountain in a zig-zag fashion, with lots of tangible tools of support, but of building an intangible spiral staircase all the way around it. It requires much backtracking and checking, because interpretations of what is happening are largely idiosyncratic and subjective. While the process is painstaking and laborious, there are infinitely fascinating and inspiring views to provide the impetus to continue.

In academic terminology we are referring to research which is qualitative because it examines qualities and entities in non-statistical ways, looking at degrees and depth rather than amounts. It is ethnographic because it involves people doing what they normally do. That is, it is not an experiment which has been specifically designed to test a particular hypothesis, nor an intervention specifically designed to create a desired outcome. It is a discourse—an idiosyncratic, individual interpretation of a set of circumstances and interactions. Because it is virtually impossible for any interpretation to be purely objective and devoid of emotional or historical bias, the discourse is subjective. Not only is it necessary to interpret other people's verbal and non-verbal interactions for the purpose of deciding on the kind of messages being communicated, but it is necessary to register some coherence for those meanings. The impossibility of specifically referring to all the interactions in the discussion of the overall study means that selection for the report of the most relevant examples is yet another subjective procedure.

Some of the problems of subjectivity in coding can be overcome by carrying out pilot analyses to test possible categories. This process allows codes to be more precisely defined, and unexpected overlaps and gaps to be examined. Essentially this kind of research is personal, and attempts to set up an ongoing discourse which allows interested people to draw their own interpretations from the data being presented and to generate new questions which might arise from it. Readers of such a discourse become co-participants in the sense that they are being invited to critique the material, and to draw from it their own personal meanings and questions.

Reports of ethnographic research usually do not propose hypotheses. The complex abstract constituents of a culture mean that long term and intensive monitoring of what happens is involved. Let us take the concept of care, a concept which is a dynamic quality of close human relationships, as an example of a complex abstract concept. In order to gain an understanding of how much care is involved in a particular relationship it is necessary to interpret messages and actions in many and varied contexts over a long period of time to be able to come to an understanding of the quality of care. Its stability is threatened by disagreement, disruption and discrepancy. The psychological, cognitive and emotional factors involved in learning are very similarly defined and influenced.

In summary, then, when mapping a culture it is necessary to look at all the factors influencing the meaning making. In an educational environment in which children are learning about mathematics the following areas need to be considered:

- Theories of learning (the developmental and cognitive characteristics of the children);
- Theories of practice (beliefs about the way in which mathematics should be taught);
- Curriculum demands (philosophy, goals, teaching techniques and evaluation procedures);
- The school culture (the way in which parental, administrative, and student bodies interpret all of the above); and
- The classroom culture (the way in which the teacher and students interpret all of the above).

It is then necessary to determine what kinds of meaning are being made in the culture. The meanings of the broad culture can be obtained from a macroscopic view encompassing its practices, people and activities while the microgenetic perspective can be derived from case studies of particular people in the practice.

How Did I Map Cultures in My Study?

Once a problem has been defined and justified, the next step is to consider the most appropriate means for gathering the first-hand information. In the case of my study, material was gathered through classroom observations using field notes, video- and audio-taped recordings in two pre-school sites in the Hunter Region of NSW, during daily half-day sessions for six weeks towards the end of the school year. Some of the children from each pre-school were followed, the next year, into their classes at school, and these children form the focus of the case studies. The school teachers volunteered to be part of the study, and mathematics lessons they took were observed during the first half of the school year. The teachers were more comfortable using me as a kind of replacement teacher or teacher aide. In one site I was able to take the same group of children each day, and to be part of the whole mathematics lesson, while in the other site, the children came to me on the verandah for their "maths group" as part of a session devoted to a variety of different kinds of activities. The lessons were video- and audiotape-recorded.

The theoretical framework, or selected findings of reputable research, provide the structure for the data analysis. Because the analysis of the discourses and the validity of interpretations of the study hinge on the solidity and appropriateness of the theoretical framework, it is important to be prepared to modify and revise original conceptions during the early stages of looking at the material.

Coding the transcript material. Before proceeding to explain the coding process in more detail, it needs to be noted that the main purpose of the process of classifying elements of linguistic interaction in such a way is to provide insights into the learning environment and the learning process. When something complex and dynamic is unravelled, examined and categorised, it is easy to "damage the data" (Richards, 1995). It is easy to become over-absorbed in the analytical process and to overinterpret what is being said, making the business of putting it all back together again for synthesising and generating logical coherence to the underlying relationships extremely difficult. It is important, therefore, to "tie up the threads" as one goes, and to keep checking back on code definitions, so that there is high coder reliability.

The main areas directly affecting the analytical framework were linguistic, mathematical, psychological and sociological.

The linguistic dimension. Young children's learning in a natural environment is very much generated by the social and physical context. Their talk is very much embedded in the situational features of an experience: language is "systematically related" to the context in which it occurs, and "is the guarantee of its significance for the learning process" (Halliday, 1975, p. 134). Halliday (1978) classified the meaning-making, or semantic, features of language according to the manner in which they were activated by the contextual, or pragmatic, features of a situation. Linguistic interactions have the potential to convey meaning in three ways: content

or informative (field), the interpersonal regulative aspects (mode), and personal attitudes, and conceptions about roles and interpersonal relationships (tenor). The most direct and explicit meanings and messages are usually conveyed through the content and socio-regulative aspects of the language, while a personal attitudes, feelings, or understandings of roles and relationships are usually conveyed less directly and more implicitly.

Children learn the linguistic, symbolic and social strategies and skills of meaning making by using language in a variety of contexts, and these experiences allow them to understand what is happening, what they should do to have their personal needs and wants fulfilled in order to feel that they belong to the social group and eventually become assimilated into a particular culture. When all the ways of making meaning in a particular socio-cultural context are combined it is known as a register of meaning making. People develop views of themselves in relation to cultural codes and mores, and these developing conceptions allow them to become equipped with the understanding, skills and strategies they need to become assimilated into the culture, and to form an identity which is culturally congruous. Access to and familiarity with the register, or linguistically conveyed meanings facilitates identity development. (For a more detailed explanation of the meaning of a register of mathematics and its relations to Halliday's functions of language, see Chapman, 1993).

The analytical framework directly relates to these three contextual forms of linguistic interaction. In its most simplified terms, the main concern of the content is with mathematical meanings, and the socio-regulative meanings form the basis for the interpretation of the sociocultural dimensions of the discourse. The psychological dimensions of motivation and affect have been incorporated into the socio-regulatory aspects of linguistic interaction. A brief explanation of each of these main dimensions follows.

Mathematical meanings. According to Bishop (1988), there are six fundamental classes of activities to be considered when describing the mathematical aspects of a culture. Bishop maintains that these "six universal activities"—which are counting, measuring, locating, designing, playing and explaining—have been found to be common across the world for meeting what might be termed "mathematically-related" needs. Moreover, mathematical ideas "may be submerged in the situational context, and the mathematical values may be overridden by different emotional or social considerations" (p. 85). There is still involvement in a mathematics culture, which is significant for the participants, but the use of mathematical concepts and the expression of mathematically-related terminology is largely unconscious at this level (Bishop, 1988).

Counting is a way of distinguishing, ordering, or quantifying objects or events abstractly or concretely. *Measuring* quantifies or distinguishes quantities or entities which cannot be counted or located. *Locating* involves positioning oneself in spatial (boundedness, continuousness, direction, time) relations to others, or other objects. *Designing* imposes an abstract or symbolic plan, structure or shape on a surface or space. *Playing* imitates or imaginatively recreates social, concrete or abstract models of reality; it is exploratory and non-goal directed, unless rules or goals are negotiated in the process of play. *Explaining* communicates verbally, the factual,

logical or evaluative aspects of ideas, questions, experiences, events or relationships between phenomena.

Motivational meanings. Motivation is being considered as a psychological construct encompassing need, desire and means. Motivation to be mathematical is about wanting to know about mathematical concepts and procedures, feeling the need to engage in mathematical activity, and believing that the means for carrying out an activity with acceptable competence is personally available. The stable operation of all three aspects constitutes a positive disposition towards becoming mathematically literate.

Malone and Lepper's (1987) model of intrinsic motivation provides the categories for analysis of motivational states. There are individual motivations of choice, challenge, curiosity, competition and imagination; interpersonal motivations are recognition, respect and co-operation. Although "control" is one of the components of individual motivation defined by Malone and Lepper (1987), this has been taken up as the major dimension of the socio-regulative behaviours and interactions. An examination of the literature on motivation in the psychological literature (see Stipek, 1993, for a detailed overview of relevant studies), in the mathematics education literature (Ellerton & Clements, 1994; Grouws & Cramer, 1989; McLeod, 1989, 1991; McLeod & Adams, 1989; Pajares, 1992; Stodolski, 1988), in the language and literacy literature (Rowe, 1994), and in sociology (Bordo, 1992; Lave & Wenger, 1991; Walkerdine, 1989, 1990; Weiner, 1994), supported and confirmed this selection of classifications.

The individual motivations are being used as one of the dimensions of analysing the activities of the practices. Imagination also forms the basis of Bishop's (1988) "playing" activity, and is therefore being incorporated into the analysis of the mathematical meanings. Curiosity is an important aspect of "explaining," because seeking an explanation is an expression of curiosity. The posing and solution of problems form part of "explaining," and are direct manifestations of the presence of cognitive challenge. Challenge and choice are cognitive, self-regulative and socio-regulative constructs as well as psychological constructs. Evidence of their functioning may therefore be less direct and explicitly expressed than the interpersonal constructs of recognition, respect and co-operation. The presence of the healthy functioning of these individual motivational constructs relates very strongly to each child's capacity and willingness to identify with the cultural setting.

Socio-regulative meanings. We are thinking about socio-regulative interactions as generating information directly and explicitly about a person's immediate wants or needs. This information is being interpreted in terms of whether the meaning is being generated in a positive way, using responsive control, or a negative way, using restrictive control. The subcategories for these two aspects of regulative interactions have been derived from varied sources: the language and literacy literature (Dore, 1984; Rowe, 1994; Snow, 1983; Wells, 1986), socio-cognitive psychology (Garton, 1992), and the early childhood literature (Stone, 1993).

Socio-regulative relations. These are the inferred, implicit or less direct and immediate aspects of expressions of wants or needs. They are the meanings affecting interpersonal relations in the long term—the meanings which accumulate to develop beliefs, attitudes and concepts about ourselves, our role in a culture,