Number sense and computation in the classroom

Kevin Jones
Lorraine Kershaw
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Number sense and computation in the classroom

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LOUISE KERSHAW
LEN SPARROW
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Preface

*Number sense and computation in the classroom* is one of a series of monographs on issues in primary mathematics education. It has been written to support initiatives in the development of student-centred learning programs at Edith Cowan University - programs designed to accommodate current views on effective adult learning strategies, as well as the development of professional and generic (key) competencies. Irrespective of the nature of individual learning programs, students and teachers working at undergraduate and postgraduate levels will find the series useful - either as a synthesis of currently available research findings and reference material, to identify potentially significant initiatives and developments in the area or as a classroom resource and guide.

The Contents and Introduction for this monograph have been designed for easy perusal, permitting rapid appraisal of its usefulness and relevance for a particular purpose. Although designed as a stand alone entity the monograph is part of a similarly titled self-contained module featuring videos/software/ print materials accommodating a wide range of teacher education programs. This independence has ensured a much wider appeal in a variety of teacher education applications.

While the content is not designed for direct utilisation as a classroom resource (these are listed separately), teachers would find the information, ideas, and examples presented, a valuable catalyst for both rationale and program construction.

Kevin Jones  
Project Leader
Acknowledgements

We firstly offer special thanks to the Committee for the Advancement of University Teaching for their strong financial and pedagogical support to produce this monograph series, and other student centred learning materials.

We greatly appreciate the contributions of the following institutions/organisations and people:

Batchelor College, Northern Territory: Kathy McMahon and Ron Stanton for putting us on the right track, and reviewing our Aboriginal mathematics education material and funding applications.

Boulder Valley Films: Merle Thornton for her generous production of our video series on mathematics learning at Yirrkala Community School.

Edith Cowan University: Nerida Ellerton for her gracious advice and direction whenever it was sought. Alistair McIntosh for reviewing our Calculator and Written and Mental Computation monographs. Paul Newhouse for reviewing our Computer monograph.

Flinders University: Christine Nicholls for sharing her wisdom and experience so generously.

University of Melbourne: Helen Watson-Verran for her commitment to the production of the Yirrkala videos.

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Yirrkala Community, Northern Territory: The Aboriginal people at Yirrkala for showing us the way.

Kevin Jones  Lorraine Kershaw  Len Sparrow

Special acknowledgement

Very special thanks to Lorraine Kershaw, our Project Office, who always had the drive to push for that extra mile. Her very substantial consulting expertise found a way through much rough terrain, and her research and writing skills were invaluable. Thanks Lorraine.

Kevin Jones
# Contents

<table>
<thead>
<tr>
<th>Preface</th>
<th>iii</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>iv</td>
</tr>
<tr>
<td><strong>CHAPTER 1</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Introduction</strong></td>
<td>1</td>
</tr>
<tr>
<td>Some questions asked</td>
<td>2</td>
</tr>
<tr>
<td>About this reading</td>
<td>3</td>
</tr>
<tr>
<td><strong>CHAPTER 2</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Everyday mathematical practices</strong></td>
<td>5</td>
</tr>
<tr>
<td>Everyday situations</td>
<td>5</td>
</tr>
<tr>
<td>Mental strategies of adults</td>
<td>10</td>
</tr>
<tr>
<td>School situations</td>
<td>12</td>
</tr>
<tr>
<td>Everyday and school methods</td>
<td>13</td>
</tr>
<tr>
<td><strong>CHAPTER 3</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Why use written computation?</strong></td>
<td>15</td>
</tr>
<tr>
<td>What is an algorithm?</td>
<td>15</td>
</tr>
<tr>
<td>Characteristics of written algorithms</td>
<td>17</td>
</tr>
<tr>
<td>Development of written algorithms</td>
<td>18</td>
</tr>
<tr>
<td>Algorithms in use</td>
<td>20</td>
</tr>
<tr>
<td>Standard written algorithms</td>
<td>22</td>
</tr>
<tr>
<td><strong>CHAPTER 4</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Number sense and mental computation</strong></td>
<td>25</td>
</tr>
<tr>
<td>Algorithms and mental computation</td>
<td>25</td>
</tr>
<tr>
<td>Number sense</td>
<td>27</td>
</tr>
<tr>
<td><strong>CHAPTER 5</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Estimation and mental computation</strong></td>
<td>31</td>
</tr>
<tr>
<td>What is meant by approximation?</td>
<td>31</td>
</tr>
<tr>
<td>Types of estimation</td>
<td>33</td>
</tr>
<tr>
<td>How is estimation useful?</td>
<td>34</td>
</tr>
<tr>
<td>Why aren’t estimation skills taught?</td>
<td>35</td>
</tr>
<tr>
<td>CHAPTER 6</td>
<td>Curriculum recommendations</td>
</tr>
<tr>
<td>-----------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td></td>
<td>Recommendations for teaching mental computation: An historical overview</td>
</tr>
<tr>
<td></td>
<td>Summary of recent recommendations</td>
</tr>
<tr>
<td>CHAPTER 7</td>
<td>Classroom implications</td>
</tr>
<tr>
<td></td>
<td>Developing skills in mental computation and estimation</td>
</tr>
<tr>
<td></td>
<td>Assessing and recording</td>
</tr>
<tr>
<td>CHAPTER 8</td>
<td>Make number sense not nonsense</td>
</tr>
<tr>
<td></td>
<td>References</td>
</tr>
<tr>
<td></td>
<td>Classroom support materials</td>
</tr>
<tr>
<td></td>
<td>Research and investigation ideas for teachers</td>
</tr>
</tbody>
</table>
Events, objects and symbols were used by ancient civilizations to communicate ideas about number quantities. Counting as a calculating strategy became inefficient particularly when mathematical explanations of the physical world were sought and changing social structures developed a need for transactions with money. Simple calculations were performed "in the head", and when larger numbers were needed, tools such as an abacus were used. Number systems and computational methods were refined to accommodate written records as these calculations became more complex. Technology now influences the type of computational methods used by people in everyday situations. Tools such as calculators and computers have reduced the need for much written computation. However, people still use many forms of mental computation in diverse situations, often making estimates rather than calculating exact answers. School practices should reflect society's computational needs.
Some questions asked

Imagine yourself at the checkout counter having selected some grocery items. The cash register total is $14.37. You reach for your two remaining ten dollar notes to pay the bill. BUT... you can only find one ten dollar note. What do you do? To return some items and have enough money to pay the bill, you could do some quick estimations mentally, use your pocket calculator for checking the results, scribble calculations on a scrap of paper, or perhaps stand there expectantly waiting for the cash register operator to carry out the necessary calculations for you.

In this situation the latter method seems most appropriate. In other situations however, different methods may be used. The choice of method could depend not only on the context of the problem but also on knowledge of number properties and operations, efficiency of the chosen method, speed of calculation, at what point an exact answer might have been needed, the reasonableness of the result and feelings of competency about solving the task successfully.

The above example has attempted to illustrate some initial questions which could be raised about the usefulness of teaching and learning various types of mental and written computation in the primary school.

These include:

✔ How do people calculate in everyday life?
✔ When are written algorithms needed?
✔ How does number sense relate to mental and written computation?
✔ Why is estimation important?
Ideas about the implications of these for classroom practice have been reflected in a number of recently published national and international statements, policies and curriculum documents e.g. *Standards for Curriculum and Evaluation in School Mathematics* (1989), *A National Statement on Mathematics for Australian Schools* (1990), *Student Outcome Statements with Indicators* (1992). Yet the reality of the classroom is that teachers largely adhere to traditional methods for teaching mental and written computation. To discard these practices teachers need sufficient background knowledge to feel confident that changes they adopt will be educationally sound, accessible and relatively simple to implement.

About this reading

In this monograph we intend to describe and discuss many differing perspectives and issues associated with this topic. The kinds of computational methods, such as mental computation, estimation and the use of technology, needed by adults in their everyday life will be identified and contrasted with those taught in schools. The usefulness of standard written algorithms will be examined and reasons advanced for their very limited application in most situations requiring computation. Developing number sense, estimation skills and utilising this knowledge in mental computation will be suggested as more appropriate for meeting children's computational needs. These aspects are reinforced by reference to recommendations made in some well known syllabus and curriculum documents from Australia and overseas.

Finally for the classroom teacher, suggestions will be made for successful implementation of these recommendations. Samples of activities are provided for the teacher to trial immediately and some suitable resources listed as a further source of classroom activities.
These ideas should provide you with a philosophical, theoretical, and practical framework upon which to base informed judgements on the planning and implementing of appropriate strategies for the teaching and learning of mental and written computation in your primary classroom.
Methods used by adults in their everyday mathematical activity have been referred to as "folk" mathematics where "problems... deal with what it will cost, how long it will take, what the score is, how much is needed" (Maier, 1980, p. 23). Mathematical calculations are carried out "for reasons and with methods different from those commonly involved in school maths" (p. 22). Adults make quick mental calculations often using invented strategies which may produce estimations rather than exact answers. A calculator may be used if the numbers are large, a number of operations are required or where an exact answer is essential. Other technological facilities such as a spreadsheet on a computer for calculating and projecting weekly expenses can be utilised.

Everyday situations

When faced with a calculation in everyday situations the computational methods chosen by adults are dependent upon many factors. These include the strategies and
number knowledge available to the person, the context of the problem, the desired outcome or reason for undertaking the task, the means available to compute and a person's feeling of competency to solve the task successfully. For example, consider the following situations.

Cooking problem:

To cook a meat dish for four using a recipe which serves 6 - 8 people. One of the ingredients is $\frac{3}{4}$ of a cup of water.

Possible methods and actions:

The first thing to do would be to halve the amounts to maintain appropriate relationships among all of the ingredients. When asked how he would halve $\frac{3}{4}$ of a cup of water, a friend said he'd estimate that $\frac{1}{2}$ of $\frac{3}{4}$ would be more than $\frac{1}{4}$ but less than $\frac{1}{2}$ a cup. "I'd just look inside the cup and estimate about where halfway was and pour in more than a quarter," he said. When questioned as to why he hadn't used his knowledge of multiplying fractions (as he had been taught at school

\[ \frac{1}{2} \times \frac{3}{4} = \frac{1 \times 3}{2 \times 4} = \frac{3}{8} \]

and a finely graded measuring cup to obtain a more accurate quantity, he laughed, "Totally unnecessary for the job at hand. The dish would taste just fine without all that bother."

Comments on the context and the methods used:

- practical procedures were selected for their appropriateness to the situation;
- some fraction knowledge was needed;
estimations were appropriate for some measures;
mental computation and approximate measures produced a satisfactory result;
paper and pencil were not needed to record any of the calculations;
paper and pencil could have been used to record the halved quantities for future reference;
school taught methods were not used.

Weekly budget problem:
A person has a weekly allowance of $130.00 to pay shared rent, food, electricity, telephone, personal clothing, entertainment and miscellaneous item costs. Rented premises are shared with 3 other people. Decisions need to be made on how much of the allowance should be allocated for each of these expenses.

Possible methods and actions:

Knowing the rent to be fixed at $100.00 per week, for example, a quick mental calculation would establish a \( \frac{1}{4} \) of the rent to be $25.00. Similar mental calculations may also be made if a fixed sum per week for food was agreed. When the electricity bill arrived a short division procedure as written computation might be used. Apportioning costs for telephone expenses may be found by using a calculator, totalling relevant itemised STD calls and summing these with a \( \frac{1}{4} \) of the telephone rental costs. To calculate the amount of money left for personal expenditure some people might make a mental estimate, some would prefer a more exact answer and use a calculator, while some could plan ahead and use the functions of a spreadsheet on a computer.
Comments on the context and the methods used:

- money problems are common everyday situations;
- knowledge of fractions, operations of addition, subtraction, multiplication and division was needed;
- mental computation and possibly some estimation were useful;
- exact answers were needed for many of the calculations;
- paper and pencil was useful for quick calculations or approximations;
- where operations such as multiplication and addition were needed for consecutive computations, the calculator could give exact answers, or be used in conjunction with mental and paper and pencil calculations, or be used as a checking device.

Both these examples show how the context of the different situations and the different purposes for the calculations affected the mathematical content knowledge needed, the choice of method and the type of calculations made.

Leisure activities

Leisure activities in which adults are engaged also provide situations where different types of computation are frequently used e.g. following a favourite sporting team like netball or football. The difference in scores of the opposing teams and the number of goals needed to win the game in a given time could be the target for some form of mental computation for an avid sporting fan—especially if the game is a close one. Occasionally paper and pencil may be used to carry out mathematical operations to calculate totals, percentages or averages. Often this material is not available and written calculations could be too laborious and time consuming. However, in the case of high profile sports which are
televised, sophisticated technological equipment provides us with all kinds of mathematical calculations represented as tables of statistics or graphs. Possible outcomes are predicted by commentators (armchair and others!) based on this information. Very little mental computation in this situation is needed by spectators. If a reasonable prediction was to be made, though, some sense would need to be made of the graphical representations and the size of the numbers in relation to the context and knowledge of the game e.g. in a cricket match to make 12 runs from the remaining 2 balls would be a very difficult task, whereas to score 12 points in a basketball game with 2 minutes to go could be an achievable task.

Tasks in employment

Calculations in everyday employment tasks must, of course, be performed accurately and efficiently. In many situations, particularly where money is concerned, computerised equipment has increased the speed and accuracy of calculations to be made as well as reducing the need to compute mentally e.g. the price of an item to be purchased is marked with a bar code, the amount is transferred electronically to the cash register which also provides the correct amount of change to be given to the customer. The only mental computation used by both the salesperson and the customer would be to count and check the change given. Many small places of business however do not have these sophisticated facilities and you will have experienced diverse computational methods, some of which are not particularly efficient (see below).

Corner deli scene:

At the corner deli three items at 45c each have been purchased. A $2.00 coin was proffered as payment. The cash register was only able to provide a record of the final
amount of the bill. The shop assistant wrote down the following:

\[
\begin{array}{c}
45 \\
\times 3 \\
\hline \\
135 \\
\end{array}
\quad
\begin{array}{c}
2.00 \\
- 1.35 \\
\hline \\
0.65 \\
\end{array}
\]

The customer meanwhile had mentally calculated three lots of 50c making $1.50 and expected to receive 50c change plus the extra three lots of 5c used in rounding up to 50 three times i.e. change of 65c.

Butcher's shop scene:

At the butcher's the shop assistant wrote down the price of each item of meat as it was weighed, summed the items and then entered the total on the cash register.

The customer had mentally rounded up or down to the nearest dollar, doubted the final figure shown on the register display and asked for the handwritten ticket to check.

**Mental strategies of adults**

How would you explain the mental strategies you might have used in each of the previous cases? If you were to compare your strategies with others you would find many variations, some of which may have been more simple than yours and less prone to error possibilities. Consider the next example and some mental strategies used by five adults to calculate a total when given this problem. How much would three drinks at 95c, $1.45 and $1.20 cost?
Explanations of their mental computation are as follows:

(i) "I'd add 95 and 45. 5+5 are 10. That's $1. 40. 40 and 20 are 60 'cos I don't have to worry about the other dollars. Then the other two dollars. That's $3.60."

(ii) "Make 95 up to $1 and $1.45 to $1.50, that's $2.50, take 10c, leaves $2.40, add $1.20 and do the dollars first because the cents will be less than a dollar, altogether that's $3.60."

(iii) "I think it's going to be about $3.50, because 95 is about $1, $1.45 is close to $1.50 and then the other dollar would make about $3.50."

(iv) "I'd use a calculator (but if you couldn't?) make 95 up to $1 and $1.45 down to $1.40, count all the dollars, that's 3 then the cents add up to 60."

(v) "I'd do it the school way, 5 + 5 = 10, put down the 0 and carry the 1, 1 and 2 are 3, then 3 add 4 is 7, and 9 more makes... wait a minute, I'm not always good at adding 9... yes, 17, put 7 down, carry 1, add the ones and that equals 3... gosh now I've forgotten how many in the tens... wait, yes I think it's three seven zero, that would be thirty seven... no, $3.70."

Only one of these methods (example v) used a procedure which is taught in schools. The procedure was long, complex and required the person to try and hold many pieces of information concurrently in short term memory. The other four methods chosen endeavoured to reduce the frequency of number manipulation and keep number combinations at a simple level.

Not only adults but children in similar everyday situations frequently use a variety of mental strategies to solve problems involving computation. For example, many children use some of the invented addition procedures illustrated above where the tens or hundreds are calculated before the ones. A typical example is Chris,
an 8 year old, who was asked how many runs he and his partner had scored in Kanga cricket.

"Well ... I hit 37 and Michael hit 24. So that's 50 ... you see 30 and 20 are 50 and 7 and 3 are 10, that's 60, and the 1 more... that makes 61." This procedure is not one which is likely to have been taught at school (Atweh, 1982).

School situations

"School maths" is mostly written computation with few mental computation activities. Written computation at school generally involves sets of examples all with the same operation:

e.g. "26 + 35, 7 + 18, 149 + 54".

Word problems are occasionally included. These usually give precise information (Maier, 1980) and state the question to be answered:

e.g. "A one litre carton of milk costs the shopkeeper 98 cents to buy from the dairy. The customer is charged $1.05. If a milk crate held 40 one litre cartons, how many crates of milk would the shopkeeper need to sell to make a total profit of $10.00?"

Children are then required to use particular procedures which they have been taught for adding, subtracting, multiplying and dividing to arrive at an exact answer. Paper and pencil are used to show each step of their computational methods.

Considerable time is spent practising how to use the taught methods with the aim of always producing an
accurate result. Exercises begin generally with a set of "sums" displaying only numbers and symbols. This is followed by some word "problems" requiring children to use the identical operation which has just been practised. Children do not have to interpret the questions to sensibly decide upon an appropriate operation.

At school, formal "mental" activities, or mental arithmetic as it is often called, are usually short speed and accuracy sessions where it is assumed that calculations are carried out "in the head" using the taught written procedures (Reys, 1984). Here instant recall of facts is emphasised. Children are expected to write the answers only to a given set of questions which often have little or no relationship to everyday situations (Maier, 1980). Testing rather than learning situations seem to be a common feature of "mental maths" sessions as questions generally test recall such as basic facts and measurement equivalents (McIntosh, 1990).

When solving problems at school children are often not given a choice, the way children and adults are in everyday situations. Choice is not available to select an appropriate tool for the task (e.g. paper and pencil, calculator, mental), a strategy, the way in which the calculation is recorded or the most relevant outcome for a particular situation (e.g. exact answer, estimation).

**Everyday and school methods**

The choice and use of particular methods used in "folk" mathematics and the time devoted to teaching these methods in school mathematics show marked differences. The results of two well known studies on the methods most adults use in their everyday non occupational tasks showed mental computation to be highly favoured as a problem solving method. Wandt and Brown (1957) found that adults said they used mental methods for 75% of their
tasks, and paper and pencil for only 25%. Another later survey reported a similar finding where more than 80% of everyday problems were solved using mental computation and estimation (Reys & Reys, 1986). A synthesis of these results and other informal surveys (McIntosh, 1990) are shown below in Table 1. Mental mathematics was favoured for 75% of the everyday mathematical tasks of adults while at school only 10% of the tasks presented to children were conducted as mental activities, usually as mental recall of facts.

Table 1: Comparison of methods used in “folk” and school mathematics

<table>
<thead>
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<th></th>
<th>Written</th>
<th>Mental</th>
<th>Calculator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Folk</td>
<td>10%</td>
<td>75%</td>
<td>15%</td>
</tr>
<tr>
<td>School</td>
<td>85%</td>
<td>10%</td>
<td>5%</td>
</tr>
</tbody>
</table>
It has been argued in the previous chapter that the nature of everyday computational practices are quite different from those often found in the classroom. To investigate further the relevance of classroom paper and pencil methods to everyday computational needs, we will:

• describe the characteristics of standard computational methods (algorithms);
• examine the usefulness of written computation as a method of calculation and as an aid to number knowledge; and
• report on some efficient and inefficient computational methods of children.

What is an algorithm?

Mathematical tasks can be carried out in a step by step procedure known as an algorithm. Algorithms can use symbols such as tally marks or numerals. They may be efficient or inefficient. For example to solve 47 - 5 in the following way would be use of an inefficient algorithm:
47 - 1 = 46
46 - 1 = 45
45 - 1 = 44
44 - 1 = 43
43 - 1 = 42

Algorithms can be mental, written or performed with a tool such as an abacus or a calculator. The procedures used by the five adults in the problem about the purchase of drinks are all examples of algorithms (see p. 11). Programmers use algorithmic procedures to direct a computer to carry out its operations.

In schools almost all algorithms are written. They are taught as standard procedures like the following decomposition method for subtraction (W.A. Syllabus, N4: P1:6, pp. 14-15):

Figure 1: Example of decomposition method of subtraction

(v) The final stage - children write the algorithm (without materials).

Develop this sequence according to ability of the class. Children’s understanding of the operation will be enhanced through these steps. Allow time for understanding of each step.

The algorithm should be developed in the context of real world problem solving.

- Possible progression of examples:

<table>
<thead>
<tr>
<th>54</th>
<th>340</th>
<th>904</th>
<th>436</th>
<th>688</th>
<th>754</th>
</tr>
</thead>
<tbody>
<tr>
<td>-32</td>
<td>-130</td>
<td>-602</td>
<td>-200</td>
<td>-346</td>
<td>-201</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>52</th>
<th>552</th>
<th>430</th>
</tr>
</thead>
<tbody>
<tr>
<td>-34</td>
<td>-226</td>
<td>-116</td>
</tr>
</tbody>
</table>

Algorithm:

State the top number first, e.g. 18 take 9.
Characteristics of written algorithms

The teaching of standard written algorithms in schools has been perceived as legitimate for many reasons. In examining these reasons Plunkett (1979, pp. 2-3) has described their characteristics in the following way. They are:

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Description</th>
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<tbody>
<tr>
<td>written</td>
<td>calculations are visible, easily able to be altered</td>
</tr>
<tr>
<td>standardised</td>
<td>everyone uses the same procedure</td>
</tr>
<tr>
<td>contracted</td>
<td>equations showing all the properties of operations can be reduced to a few lines of symbols</td>
</tr>
<tr>
<td>efficient</td>
<td>e.g. if the ones are totalled first in addition, no subsequent alterations need to be made to the units column when the addition of the tens and hundreds has been completed</td>
</tr>
<tr>
<td>symbolic</td>
<td>only symbols are used, usually without accompanying explanations</td>
</tr>
<tr>
<td>general</td>
<td>they will always work with all kinds of numbers e.g. small or large, decimals</td>
</tr>
<tr>
<td>analytic</td>
<td>e.g. digits are calculated separately to tens</td>
</tr>
<tr>
<td>not easily internalised</td>
<td>they don't tend to reflect the way people think about numbers</td>
</tr>
<tr>
<td>not conducive to understanding</td>
<td>methods and rules are adhered to without thinking</td>
</tr>
<tr>
<td>traditional</td>
<td>the number rules are taught rather than the concept and ideas about the operation</td>
</tr>
</tbody>
</table>
At first glance it may seem that standard written algorithms have many advantages, and that they are and always will be useful. On the contrary history and research shows that not only did their present form recently emerge in response to cultural and economic changes in our society but also their use in our present day society has been found to inhibit efficient computation needed in everyday situations (Hope, 1986; Nunes, 1992).

The following brief summary of the historical development of written algorithms shows that computation strategies have evolved over time as needs changed and systems were invented to meet those needs.

Development of written algorithms

In ancient Babylon, prior to 2000 B.C. and long before money was invented or paper made, symbols representing quantities of objects were used to aid counting. The Mayan system of notation used symbols which were written underneath one another rather than alongside each other like many other cultures. The Greeks originally used letters for numbers and gradually joined some of these together to form larger numbers. These larger numbers assisted a small minority of people such as shopkeepers who needed to make calculations when goods were being purchased. Written calculation systems became increasingly necessary as trade grew, the barter system declined, money came into use and many more people needed to record their transactions accurately. No longer was oral counting and the available notation systems sufficient to cater for computational needs. In addition, memory load became a problem as more complex calculations needed to be made (Nunes, 1992).

Not all the numeration systems lent themselves readily to the use of written algorithms. The Roman system is one
where different letters are written alongside one another and then mentally added to calculate the value of the symbols. If you have ever tried to perform a written calculation using Roman numerals, though, you will find it an extremely difficult and cumbersome task! They used an abacus instead.

The Hindu-Arabic base ten system, which is used in our Western culture, made complex calculations easier to record and interpret as a written procedure. Thus a usable written algorithms was developed. This system makes use of ten symbols, with zero as a place holder. The position of each symbol in any written number denotes its value e.g. the number one thousand and seventy one in symbolic form would be 1071, where the 0 indicates there are no hundreds, the 1 on the far left shows one thousand and the 1 on the right has a value of one unit.

The form which standard written algorithms now take has not always been fashionable. Some three or four decades ago the “borrow” and “pay back” method was used for subtraction. In the middle ages subtraction was carried out using a "crossing out" method (McIntosh, 1990).

Figure 2: "Crossing out" method of subtraction

\[
\begin{array}{c}
25 \\
\hline
26 4 \\
\hline
4 2 3 \\
\hline
423 - 169 = 254
\end{array}
\]
Of the standard written algorithms found in schools today, decomposition is considered to be the best strategy to use for subtraction. It was felt that this method embodies understanding of number when taught through the use of materials such as Multibase Arithmetic Blocks (MABs) (W.A. Syllabus, 1989). These school taught strategies have often been found to be unhelpful and detrimental to efficient computation in many everyday situations.

Algorithms in use

Results of a 1983 National Assessment of Performance Unit (APU) test conducted in the U.K. with some young adults (Hope, 1986) revealed that

"when asked to multiply numbers like 90 and 70 (mentally) only 55% of the seventeen year olds were able to do so...55% were unable to calculate an exercise like 4 x 625 mentally... almost 40% were unable to complete solutions to items similar to 3500/35 within a ten second time limit" (pp. 47-48).

The question to ask here is why were so many of these young adults so poor at simple mental calculations? Hope (1986) maintained that rigid adoption of the school taught rules for written algorithms influenced children and adults in their choice and ability to use efficient computational methods. He cited the following examples of a child, an unskilled calculator, who chose to use the following written methods for some simple calculations which could have been carried out more quickly in her head:
Hope (1986) found that skilled mental calculators also used algorithms, but these were not standard procedures. They used their knowledge of the number system and invented algorithms to arrive at a quick, accurate solution.

For example, one young child when given some tasks explained the procedures used in the following way:

"8 x 99
1 did 800 - 8 = 792
25 x 480
Well, 25 is 100 divided by 4. So I divided 480 by 4 to get 120 and multiplied that by 100."

(Hope, 1986, p. 52)

The context of the task has also been found to strongly influence the efficiency of children's calculations (Carraher, Carraher & Schliemann, 1985; Shuard, 1986). Carraher et al (1985) carried out a study with a group of Brazilian children aged 9-15 years. The children worked out of school hours selling goods in street markets. They were observed in the market place whilst making calculations in their accustomed way. In addition they
were formally tested as they used pencil and paper to solve some problems based on the market place context. The researchers found that context-embedded problems were more easily solved, and invented mental strategies were chosen and used successfully in the market place. When paper and pencil were used in simulated market place problems, school-type symbols and routines interfered with the solving process.

In the U. K. a study on the mathematical performance of a group of 11 year olds reported similar findings (Shuard, 1986). Only one-quarter were able to correctly answer a question on the measurement of length involving addition of fractions in a paper and pencil test. However when they were given a piece of string to use as an aid in making fraction lengths, 42% were successful. Like the previous market place example children were far more successful when the problem was presented in a practical situation and they did not use standard algorithms.

**Standard written algorithms**

It is clear that written computational strategies, which are taught as standard procedures in schools, are not particularly useful in our society today. By continuing to place such emphasis on children knowing about and practising standard written algorithms teachers are not responding to children's needs and the nature of everyday situations. There also seems to be a belief by teachers that if children learn the standard algorithm they will understand number concepts, and that children's numeracy skills can be measured by their competency in using standard written methods. As Duffin (1991, p. 41) said teachers “seem to have the idea that these algorithms are basic to number experience and must therefore still be taught in spite of technological developments which seem to make them redundant.”
Many other reasons have been advanced to support the argument that the teaching of standard algorithms should no longer be included in the mathematics curriculum. Some powerful and convincing reasons are summarised below:

✓ When presented with problems requiring a written method children have to think more about how to set out the procedures than the reasons for choosing to use them.

✓ Children spend time practising the methods rather than developing an understanding of the mathematics need to solve problems.

✓ Children work in isolation without discussion, the opportunity to share ideas or actively develop their understanding of numbers as this is considered to be “cheating”.

✓ Little understanding of the number system and number properties is gained. Number relationships are not used.

✓ Creative or inventive thinking is discouraged or prevented.

✓ By emphasising standard procedures of written algorithms, ability to create mental strategies may be hampered.

✓ Negative attitudes to mathematics can occur as it is perceived as consisting of long and tedious pages of “sums”- mainly written computational examples.

✓ Time is used inefficiently, as examples are generally copied from workbook or board and lengthy operations such as addition with four addends of two or more digit numbers are calculated.

✓ Everyday situations make limited use of paper and pencil methods.

✓ Reasonableness of solutions are not checked, algorithmic procedures are. Children seem to believe that solutions reached in this way are correct.

(Reys, 1984; Hope, 1987; Jones, 1988; Sowder & Sowder, 1989)
24 • Number sense and computation in the classroom
The practical situations described in the previous sections highlighted the growing conviction that where choice was available mental computation was favoured above written computation. One of the major reasons for advancing the argument that children's mental computation skills should be fostered, has been that number sense will be developed, and used with reason to engage in any form of computation (Hope & Sherrill, 1987; Sowder, 1992a). Before discussing what is meant by number sense it is useful to explain the nature of mental algorithms.

Algorithms and mental computation

Mental computation has been defined by Reys (1992) as:

"the process of carrying out arithmetic calculations without the aid of calculators or external recording devices" (p. 4).
Characteristics of mental algorithms according to Plunkett (1978) could be described as:

- transitory and often difficult to explain;
- using many different methods successfully to reach the same solution;
- adaptable to simple or complex calculations;
- unique to the user who chooses method and organises calculations in an individual way;
- not intended to be written as step by step procedures;
- reflecting current level of number understanding;
- useful in arriving at an approximation before or instead of an accurate answer; and
- difficult to use with complex operations or large numbers.

This description of the nature of mental computation is useful in examining the methods used by children. The next two problems and some children's responses indicate evidence of many of the above characteristics.

We asked a large group of 10 year olds to give us a written explanation of how they calculated mentally 48 add 9. Their recording included the following:

"8 and 8 are 16 and 1 more is 17, add 40, that makes 57"
"48 and 10 are 58, take 1, 57"
"9 and 1 is 10, add 48, 58, take 1"
"48 and 2 more is 50, then 7 is 57"

and this interesting response:

"8 and 8 are 17 and that's 57"

A child (aged about 7 years) who was familiar with paper and pencil methods and summing 2 digit numbers was
asked to solve $246 + 178$ mentally. Her reply showed some of the features of mental computation listed above, particularly her knowledge of numbers and invented procedures.

Well, 2 plus 1 is 3, so I know it's 200 and 100, so now it's somewhere in the three hundreds. And then you have to add the tens on. And the tens are 4 and 7... Well, um. If you started at 70, 80, 90, 100. Right? And that's four hundreds. So now you're already in the three hundreds because of the [100 + 200], but now you're in the four hundreds because of that [40 + 70]. But you've got one more ten. So if you're doing it 300 + 40 plus 70, you'd have 410. But you're not doing that. So what you need to do then is add 6 more onto 10, which is 16. And then 8 more: 17, 18, 19, 20, 21, 22, 23, 24. So that's 124. I mean 424.

(Carpenter, 1989, p. 90)

The number ideas and the ways in which number relationships were used in the previous two examples also show evidence of what has recently been referred to as number sense.

**Number sense**

Number sense has been described in many ways:

- as an aspect of commonsense (Sowder & Sowder, 1989).
- as being linked to higher order thinking and reasoning (Resnick, 1989).
- in metacognitive terms e.g. monitoring knowledge of number pairs (Silver, 1989).
Ideas about number sense are not simple to grasp as theorists have only recently begun to try and explore what is really meant by having and using number knowledge. It is not clear exactly what that knowledge is nor how it is managed and used. Certainly there seems to be a general consensus that number sense is concerned with thinking about number, being creative with numbers, developing organisational processes which help to relate ideas about numbers, and their reasonableness in different contexts (Howden, 1989).

Some examples which would indicate children have number sense abilities are listed below:

1. A young child thinking of 45c as three 10c and three 5c to share with two other friends - thinking about number relationships relevant to a context.

2. Knowing that the difference between 6 and 9 is the same as the difference between 436 and 439.

3. Knowing that 1000 marbles wouldn't fit in the jam jar - the reasonableness of the magnitude of the number in relation to the context.

4. Knowing that items costing 85c and $1.05 respectively are each close to $1 and so the total will be about $2.00 - using estimation to check reasonableness of result.
These ideas and examples about number sense clearly highlight the significance of understanding numbers, their composition, and relationships in deriving reasonable solutions to problems. For example, consider the problem of finding the difference between $1.9$ and $3.6$. A child who demonstrated number sense ability said that the solution would be about $1.5$. She mentally made the $1.9$ up to $2.0$, said the difference was now $1.6$, added on the $0.1$ and gave an answer of $1.7$. Another child when presented with the same problem said she had a mental picture of the $1.9$ sitting below the $3.6$ with the decimal points lined up. She then proceeded to explain how she had used the decomposition method of subtraction to arrive at a solution of $1.7$. When asked to explain or justify her answer she said she checked her mental algorithm. Both girls provided a correct solution but the second girl did not show as flexible an understanding of numbers and their relationships as evidenced by her method of checking her answer.

The way in which the first girl solved the problem illustrates how number sense, mental computation and
estimation were used together, quickly and successfully. Estimation and approximation skills are particularly important in being able to manipulate numbers mentally.
It is not obvious from this statement how estimation and approximation differ. Sowder (1992a) cites Thompson’s distinction where estimation relates to “number of objects in a collection (or) the result of a numerical calculation” (p.373) and approximation is concerned with attempting to get close to an exact value. Many opinions exist, however, about the differences between the two, with some researchers suggesting the terms are interchangeable (Sowder, 1992a).

What is meant by approximation?

Most would agree that approximations are used in at least two distinct ways, namely in measuring and in arithmetic calculations (Hilton & Pedersen, 1986). Units of measure are all approximations. The smaller the unit becomes the more accurate the measurement is likely to be, but it could
never be exact. Particular situations determine the type of refined measurement needed. The hundredths of a second which are used as units of measure in world class athletics, for example, are needed to make fine distinctions among the top placegetters in an event. The measurement is as close an approximation as is possible given the capabilities of the technological equipment in use. It is not exact, unlike the irrefutable answer of four, to the question “What do two and two make?” Unless perhaps you have asked a creative accountant who replies, “What would you like it to be!”

One way to make numerical calculations easier is to make an approximation by changing the numbers to a more manageable form, e.g. rounding 273 and 59 to 300 and 50 respectively to sum them, providing 350 as an estimate of the total. Approximation procedures were used in arriving at an estimated result which was considered sufficiently accurate for that particular situation. We are all confronted with examples of approximations everyday. For example, now that one and two cent coins are no longer being used cash registers make the necessary additions and are often programmed to calculate the rounded figure, as an approximation procedure. Where this facility is not available operators are provided with a schedule which shows the agreed rounding patterns to be used. Many of us would also mentally round up or down to the nearest dollar the cost of each item to keep a running total as we push the trolley around the supermarket. Again we would be using certain rounding rules to obtain a reasonable total. In summary then, approximation could be seen as the processes of measuring and rounding.

While much more could be written about approximation in relation to measurement and approximation processes, the importance here is the ability to use procedures in mental computation activities which give rise to estimates.
Types of estimation

According to Sowder (1992a), estimation has three forms, namely, computational, measurement and numerosity.

Computational estimation refers to the result obtained by choosing or inventing strategies which allow the person to perform mental calculations needed to solve a problem. Studies investigating the characteristics of estimators found that good estimators changed the numbers in some way to make calculations easier, often did not use school taught algorithms, calculated from left to right, and showed number sense abilities when using varied strategies. In addition they seemed confident of their mental computation ability and were generally unperturbed by mistakes (Hope & Sherrill, 1987; Sowder, 1992a).

Measurement estimation is concerned with making an estimate, and sometimes a calculation, about a characteristic of an object, such as estimating the number of glasses of cordial in a jug. In this example the person needs to have had considerable practical experience with the unit measure, that is, the standard measures of litre and millilitre or the glass as a non standard unit, to use as a referent so that the result will be reasonable.

Numerosity estimation as the name implies involves numerical calculations which are often connected to measurement problems. “Guess the number of jelly beans in the jar” is a typical example. Good estimators might try to estimate the number of jelly beans in one layer by approximation (e.g. by averaging the number of jelly beans in the top and bottom layers), approximate the number of layers (e.g. by counting) and multiply the two to find an estimated result. In this example averaging the number of jelly beans and counting roughly the number
of layers are seen as rounding, and estimators would therefore be using their approximation skills.

How is estimation useful?

We have seen how approximation is an integral part of measurement experiences which form a very large part of our everyday mathematical activity. We have argued that estimation and mental computation are interwoven with the development and application of number sense and explained how number sense aids mental calculation. We have also given some examples which illustrated the usefulness of estimation in a broad range of everyday tasks. It is appropriate now to summarise these and later to review some curriculum recommendations which support its use in the school environment.

Usiskin (1986) included the following suggestions in his discussion about the usefulness of estimation in everyday life where:

✓ values may not be known or vary, e.g. estimating how far your wages will stretch;
✓ past events or future trends provide an understanding of social groups and aid in predictions;
✓ understanding is facilitated, e.g. large numbers may be rounded to aid interpretation of population distribution: in this case exactness would make understanding difficult;
✓ calculations are made simple e.g. by rounding the cost of a $5.79 item to $6.00 to calculate how many 8 of them would cost; and
✓ estimates allow comparisons to be made over time if those estimates are calculated in a consistent way, e.g. rounding petrol costs to estimate a car's fuel consumption over a four week period.
Why aren't estimation skills taught?

If estimation and mental computation are useful tools in so many different everyday activities, and in contributing to the development of number sense, we should query why experiences involving estimation are not part of the regular mathematics curriculum.

Many reasons have been suggested (Trafton, 1986; Usiskin, 1986; Bana, 1990; Sowder, 1992a). Traditionally mathematics has been seen as an exact science which always produces one correct, precise answer to any problem. To make an estimate is to produce varying answers all of which could be acceptable. This would necessitate a change in teaching style where it was easier to mark a set of correct answers and not have to listen to various responses. Children's expectations would need changing and therefore so would their calculation methods, as these would no longer be duplicates of "correct" algorithmic procedures.

Children are fearful of making estimations which are to be used as predictions. This could be for the reasons stated above or because children are unfamiliar with the objects or events about which they have to make estimates.

Sowder (1992a) cited a number of research studies which found that people's estimation skills developed with experience and maturity. She suggested that when mental computation included estimation the process of deriving a solution became quite complex. Skills and strategies involved in estimation are extremely varied due to the many differing situations in which they are used. According to Sowder their effective use can therefore depend on children's everyday and numerical experiences as well as their cognitive development. In addition well organised short term memory seems to be required (Howe & Ceci, 1979; Reys, 1985). If children need to possess this
array of skills and abilities to be good estimators it is not surprising that the task of teaching estimation is seen as too difficult and beyond the capability of many primary school children.

A further reason for the exclusion of estimation activities in mathematics programs is that teachers are unsure how to assess skill in estimation. If teachers find this difficult or if few guidelines exist for gradually assisting children to acquire estimation skills then they are unlikely to pay much attention to this aspect of mathematics.

Recent statements and curriculum documents in Australia and overseas have attempted to address these difficulties by explaining why estimation and mental computation are important in the learning of mathematics.

Carroll (1992) has listed some supporting reasons:

- Efficiency.
- Provides insight into student thinking and understanding.
- Encourages thoughtful inspection before doing the computation.
- Promotes the use of natural basic mathematical properties.
- Utilises visual thinking skills.
- Develops mathematical reasoning.
- Stimulates, encourages and rewards pattern explorations.

(pp. 4-5)
Before reviewing any curriculum documents it would be useful to outline the historical context of the fluctuating emphases on teaching mental computation. Together with the reasons advanced in the previous chapters this should provide a basis for understanding the recommendations which have been made in this and other countries for the teaching and learning of mental and written computation.

Recommendations for teaching mental computation: An historical overview

Reys (1985) provides a brief but informative overview of the ways in which ideas about mental computation have changed over the last 150 years. In the mathematics curriculum of the mid 1800s mental arithmetic instruction (as it was named) was considered important in disciplining the mind through practising lots of examples which demanded quick recall of facts. It had in fact begun to rival and even replace the learning of Latin as a mental discipline. Mental tasks became more complex as the number of steps needed to solve the problem increased. At the beginning of the twentieth century there were
strong moves against these practices as they were deemed unreasonable in their expectations and provided children with little mathematical benefits. Written computation became important. However some three or four decades later concern was expressed about the dependency of children on written computation methods to solve all problems. In recent years educators have argued that children at school should be taught the mental computation methods that are used in everyday activities outside the school. The 1980s and 90s have seen the emergence of ideas about number sense and the ways in which mental computation activities can help children learn and understand mathematics.

These latter ideas have provided the basis for the development of curriculum recommendations made by state and national education organisations.

Summary of recent recommendations

The five documents listed below were chosen for this summary because they represent current thinking of mathematics educators in Western Australia, Australia, the U.K. and the U.S.A. Main points only are mentioned. The documents themselves will obviously provide a far more comprehensive picture of rationales, aims, objectives and teaching perspectives.

1. Learning mathematics handbook: Pre-primary to stage seven mathematics syllabus
   (Curriculum Programmes Branch, 1989)

   Written computation should be de-emphasised and mental computation given a higher profile to aid children's understanding of number. Activities which reflect everyday out of school practices are suggested. This would entail teachers designing realistic and meaningful mental and written computation tasks which
had relevance for children. In particular it recommended that the teaching of complex written algorithms such as long division be omitted from the mathematics program.

2. **Curriculum and Evaluation Standards in School Mathematics**  
   *(National Council of Teachers of Mathematics, 1989)*

   In recognising the changes in the mathematical needs of students over the last few decades this document emphasises the importance of mathematical literacy. Ideas about mathematical literacy are explained as a set of curriculum and evaluation standards for students at different levels. There is a focus on number sense. Students are expected to be able to make informed decisions about what tools and methods to choose for calculating, and to judge the reasonableness of their results. Recognition of situations which require exact answers or estimates, the efficient use of mental computation strategies and simple written algorithms, and the ability to use a calculator and a computer are considered necessary for children to develop mathematical literacy.

3. **A national statement on mathematics for Australian schools**  
   *(Australian Education Council, 1990)*

   Children should have confidence to choose an appropriate tool for the computational task at hand and be able to use strategies that will produce sensible answers which have the degree of accuracy required for the task. Tools could include mental computation, paper and pencil, calculators and computers. The teaching approach should encourage children to use approximations and estimates which assist them to develop a sound understanding of place value and the number system. Children's use of personal and invented strategies for mental computation is to be
supported. Finally it is recommended that children have access to calculators and computers when required for computational purposes, and that paper and pencil be used in conjunction with these and mental computation.

4. Mathematics profiles
(Australian Education Council, 1992, April)

A series of mathematical achievement statements which relate to different primary school levels are listed. The sections on number and measurement make particular reference to mental computation and estimation. An excerpt from Level 5, illustrates this:

At Level 5 (number) students can:...

Recognise occasions where it is appropriate to approximate and choose, use and explain techniques for estimating numbers...

devise and describing strategies...

use approximation to check calculations...

form reasonable estimates...

(p. 10)

5. Mathematics in the National Curriculum
(Department of Education and Science, 1991)

Attainment targets in mathematics (U.K.) are detailed as explanatory statements supported by examples for a range of levels. “Pupils doing calculations” in the Non Statutory Guidance section recommends that pupils should:

• be encouraged to use their own methods for written and mental computations;

• use mental computation before other methods;
• use methods which promote their understanding of number;
• make use of the calculator for speed of computation and understanding of number; and
• be able to make approximations, estimations and judge reasonableness of answers.
Implications for classroom practice are most usefully stated as suggestions for teachers. In view of the decreased need for written computation, suggestions will mostly focus on mental computation and estimation. They have been framed as questions in the following way to enable teachers to appraise the nature of the mental mathematics activities which they are planning to implement.

Are mental computation and estimation activities:

- chosen because the skills are used in everyday mathematical problems?
- encouraged in everyday mathematical problems?
- integrated with everyday mathematical tasks?
- relevant to children's mathematical needs?
- relevant to the mathematical topics currently being studied?
- interesting and enjoyable for all children?
It is clear from an analysis of these questions that many are concerned with children’s feelings. It is critically important that children feel confident about tackling mental computation tasks and discussing their use of strategies. This means that an atmosphere of trust must be developed in the classroom where the teacher values children’s attempts and responds to them in a non-judgemental way.

In addition to creating this type of supportive environment the teacher needs to:

- provide a range of problem contexts where some require exact solutions and others only estimates;
- analyse carefully textbook word problems to determine their realism;
- utilise problem situations in which children find themselves;
Reasonableness of responses is a critical feature of mental computation activities and children need to develop some appropriate strategies. Garofalo and Bryant (1992) make the following teaching suggestions:

1. Give students a wide variety of problems (which) include one-step, multistep, and nonroutine problems...

2. Facilitate students' discussion and interpretations of the meaning and importance of problem conditions. ...make sense of the problem situation...

3. Encourage students to estimate answers before carrying out calculations...

4. ...(ask) them to judge whether their answers “make sense”...

(Maier 1980; Trafton, 1986; Sowder, 1992a)
Developing skills in mental computation and estimation

There are numerous references in the literature to specific teaching ideas which would assist children to develop and use efficient mental computation strategies and estimation skills as well as encourage children to use them (e.g. Rathmell, 1978; Trafton, 1978; McIntosh, 1980; Trafton, 1986; McIntosh, 1988; McIntosh, 1990; Parker & Widmer, 1992; Greenes et al, 1993). It is beyond the scope of this reading to provide explanations and examples of all of these ideas. A selection only will be outlined in the sections on mental computation activities, developing estimation skills, and strategies for basic fact recall. Two specific techniques will also be explained, including one where children have to utilise the spreadsheet functions of a computer. Further reading of the above will provide more detail on each of the activities summarised below, while other references will give a wider range of activities for teachers to use.

Ideas for mental computation activities
(McIntosh, 1980; McIntosh, 1988)

The activities listed below are suitable for most Year levels. Sessions of no more than 15 minutes are recommended, although in our own trailing of some of these ideas we found some of the older children wanted initially to prolong the sessions—due possibly to their novelty!

1. Today’s number is ...

The teacher writes a number on the board. No paper or pencils are allowed. The children provide numerical expressions which when solved would give the teacher’s blackboarded number as the answer. The teacher
blackboards every response in categories and in column form. As each response is written on the board the teacher asks the class to check it for accuracy and if necessary corrections are made by the children. Patterns from these can be discussed, questions asked and other response formats requested by the teacher.

Example (our trial with Year 7 children):

<table>
<thead>
<tr>
<th>Teacher: Today's number is 15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selection only of blackboarded responses:</td>
</tr>
<tr>
<td>11 + 4</td>
</tr>
<tr>
<td>9 + 6</td>
</tr>
<tr>
<td>8 + 7</td>
</tr>
<tr>
<td>(15 - 20) / 2</td>
</tr>
<tr>
<td>(2 x 5) + (2 x 2) + (1 x 1)</td>
</tr>
<tr>
<td>(30 - 10) - 5</td>
</tr>
</tbody>
</table>

As responses were written the teacher occasionally asked questions like:

“What if I put - 45 in the second column? Could we find what to take it away from by looking for a pattern and perhaps using some of the information in the first column?”

2. Sum story

The teacher writes a numerical expression on the board. Children are then asked to make up a real life story which will make sense and then describe it to the class. The teacher could give the first story as an example. About five stories from different children could be accepted before another expression is given. Stories may be three or four sentences long.
Example (Year 2):

Blackboard expression: 10 - 7

Teacher story: Only 3 kids in our team of 10 got a home run, 7 didn’t.

Note the following points about this activity:

- The teacher’s story modelled the everyday language of young children;
- The information was provided in a story form;
- No “≠” sign was used, as the intention was to move away from the traditional “number sentence” format;
- No question was asked;
- No “answer” was expected, as an understanding of the expression was displayed in the story.

3. Find my number

The teacher thinks of a number within given parameters. Children then ask questions of the teacher who may only answer Yes or No. Their questions are blackboarded. Answers are likely to use number concepts such as place value, multiple, prime and composite. The value of writing responses on the board for all to see is that children can use this information to mentally check and eliminate possible answers. It also enables the teacher to generate discussion about predicting the answer and so engage the children in thinking about number ideas.
Developing estimation skills
(Trafton, 1978)

Three interesting approaches have been selected for explanation. The teacher should design problems with different contexts and emphasise the approach which would be most suitable in each case.

1. Rounding and computation

Computations are carried out using any of the four operations by rounding numbers e.g. to the nearest ten, hundred or thousand. Children need to learn a consistent rule for rounding from a mid point. Multiplication is suggested as a good beginning because it is felt that most children would still be too concerned with obtaining an exact answer to familiar written addition and subtraction problems. One particular type of problem suggested was where the most appropriate estimate had to be chosen from a given selection of estimates to teacher posed questions.

2. Reference points

Problems which deal with money provide an everyday context and are one way of using reference points for estimation. For example, children can be asked about how many items they could purchase with a given sum of money.

3. Estimating ranges

These types of estimation problems involve the use of bracketing where as a whole group the children have the opportunity to estimate aspects of measurement and number e.g. estimate the time taken for a task or the number of objects in a container. The procedure is as follows. The task is given and individual children are
asked to provide an upper and a lower limit. All responses are written on the blackboard. Other responses within these limits are discussed and accepted until the children decide that the final answer could not be more than a certain amount or less than a certain amount. This narrowing of the range of reasonable answers gives children the chance to refine their estimates. The last two numbers of the range are then averaged to produce a final estimate. The children's estimation strategies are then discussed and evaluated. (McIntosh, 1980)

Example (our trial with Year 7 children):

<table>
<thead>
<tr>
<th>Couldn't be less than</th>
<th>Couldn't be more than</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 seconds</td>
<td>40</td>
</tr>
<tr>
<td>12</td>
<td>35</td>
</tr>
<tr>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>19</td>
<td>25</td>
</tr>
</tbody>
</table>

Children decided that the range could not be refined any further and the average of 19 and 25 was calculated. The actual time taken had been calculated by the teacher, using a stopwatch, at 19 seconds.
Strategies for basic fact recall
(Rathmell, 1978)

A sequence of steps is outlined for teaching thinking strategies which could aid children's recall of basic addition, subtraction, multiplication and division facts. Some of the multiplication strategies have been selected to illustrate the range of possible strategies which are used by many people and could be useful for children.

1. Skip counting
Here the problem might be:

"What is the answer to 6 x 3?"

The child counts mentally by threes - 3, 6, 9, 12, 15, 18 to find the answer.

2. Repeated addition
A multiplication fact such as 6 x 4 is not known but could be solved by mentally thinking of the problem as 6 + 6 = 12, 12 + 12 is 24.

3. Using known facts
A known fact is used to arrive at the answer to an unknown fact e.g. 8 x 6 is not known but 6 x 6 is known. Two times six (12) is added to 36 to give the answer of 48.

4. Using patterns
The nine times table is a good example. This pattern can be helpful as the sum of the digits in the answer must total nine e.g. 7 x 9 is less than 7 x 10 = 70, this gives a 6 as the first digit of the answer, the total of the digits must be 9 and so the result is 63.
Two specific techniques
(Parker & Widmer, 1992; Greenes et al, 1993)

1. T-E-M-T-T (trial, error, modified trial)

The approach suggested by Parker & Widmer (1992) is called the T-E-M-T-T, trial, error, modified trial through technology. Spreadsheets on a computer can be used to enter initial data from which estimates are made and trailed to obtain the result of a future outcome. For example, supposing Susie was saving up a proportion of her pocket money to buy a video game which cost $23.50. The problem would be to estimate and then determine how many weeks she would take to save the required amount of money. The estimated result would be checked using the spreadsheet facilities of a computer.

2. Fit the facts

Fit the Facts is an approach suggested by Greenes et al (1993) to assist children to develop and use number sense skills. A short story, in a context familiar to the children, is provided, with all the numerical data removed but listed in an adjacent table. Children have to make decisions about which numbers from the table best fit in the missing spaces. They record their thinking steps as they make decisions. This procedure illustrates their use of number relationships and properties to justify mathematically and contextually appropriate answers.

As can be seen from the design of many of the activities described above, children of varying abilities can all make successful contributions to mental computation activities. No child need be threatened in any way by having to produce an immediate, accurate response, yet all can be engaged in computing mentally. Discussion and sharing of strategies in a supportive environment is a feature of all good estimation and mental computation activities.
Assessing and recording

Children's discussion and their responses to challenging and open-ended questions during mental computation activities can provide the teacher with information on which to base an assessment of children's learning and understanding.

Another assessment technique is to provide the children with a mental computation task and upon its completion ask them to write down what procedures they used to solve the problem, that is, record what they were thinking. The Fit the Facts approach described previously is a good example, where children recorded their thinking steps to justify the decisions they made. Younger children could write a story, draw pictures or combine these to explain their strategies. Older children could decide upon their own format to communicate the strategies they used in a mental computation task. An evaluation of these types of records could provide the teacher with some indication of children's number sense (McIntosh, 1988).

Reys (1985) suggested conducting mental computation tests on a periodic basis. He advised limiting the number of items in the test to a maximum of 20. These items should mostly focus on one aspect of mathematics such as addition, with an increase in diversity as children became more proficient. Timing of test items was important to emphasise the mental nature of the tests. If too much time is given for each item then quick estimates and mental calculations would not be undertaken. Variety in format was suggested, for example, read word problems, use the overhead projector to show a problem for a few seconds only. Try some "nested" problems which are spaced out in the test e.g. $3 \times 7$, $3 \times 70$, $30 \times 70$; $4 \times 100$ and $4 \times 99$ to show children's understanding of number relationships. This idea can also be used in a future teaching situation.
What is being suggested here is that mental computation tests should be used sparingly, unlike the daily 10 minute speed and accuracy “testing” situations which go by the name of mental mathematics activities and still occur in most classrooms today. Their format should be varied but purposeful for the children as well as the teacher. The testing method adopted must be clearly compatible with the testing objectives.
It is not surprising that the curriculum recommendations outlined in this monograph were all made within the last four years, given the ground swell of opinion on the inappropriateness of school practices in mental and written computation. The impact of the findings of numerous research studies and the recommendations of noteworthy curriculum statements on current practices are considerable. It now seems fair to say that the relative time spent in school on mental and written computation activities is not appropriate to children's needs. Devoting large amounts of time to the teaching of standard written algorithms and conducting the typical "mental" sessions of old have not served any useful purpose in assisting children to understand and apply number ideas. Practices in classrooms must be changed.

Children must be allowed to decide what computational methods meet the demands of the tasks in which they are engaged. This means that children must feel confident in using a range of methods (such as estimation skills and mental computation strategies) and tools (such as the calculator, computer, and paper and pencil). The teacher's responsibility is to provide suitable mathematical
experiences - experiences which offer children choice and support personal inventiveness.
References


reasonableness: Some observations and suggestions. *Arithmetic Teacher*, 210-212.


Hope, J.A. (1986). Mental calculation: Anachronism or basic skill? In H.L. Schoen & M.J. Zweng (Eds.), *Estimation and mental computation*. (pp. 45-54). Reston, Virginia: NCTM.


Classroom support materials


# Research and investigation ideas for teachers

- Survey adults to discover their mental methods for calculating in everyday situations e.g. 6 stamps at 45c each OR their change from a $5 note.

- Ask children to show you different ways to do a calculation e.g. 48 x 12.

- Ask people to explain how and why their calculating method works (not how to do the problem).

- Find out what people’s memories are of doing school calculations, both mental and written.
About the authors

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Other titles in the series
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